Structural properties of Petri nets, calculating invariants

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Recall: Dynamic properties

- Example: Model of a workflow (tasks + activities + resources)
- Properties analyzed
  - Does the system halt? Deadlock
  - Can certain activities be performed? Liveness
  - Do tasks overwhelm? Boundedness
  - Can we return to the initial state? Reversibility
  - Is there a processing loop? Home state
  - Can activities be stopped? Persistence
  - Is there an activity lacking resources? Fairness
- Problem: Exploring a large state space

Recall: Analysis methods

Depth of the analysis:

- Simulation ⇐ Traverse single trajectiories
- Full exploration of the state space ⇐ Traverse all trajectories from a given initial state (exhaustive traversal)
  - Analysis of reachability graph: Dynamic (behavioral) properties
  - Model checking
- Analysis of the net structure ⇐ Properties independent from the initial state (hold for every initial state)
  - Static analysis: Structural properties
  - Invariant analysis

Main idea of structural analysis

- Can we state something without traversing / exploring the state space?
  - Based only on the structure (places, transitions, arcs)
  - Analysis independent from the initial state
  - In certain cases only approximate results!
- Approximate analysis is safe if it covers the real behavior
  - If no counterexample is found for the examined property (erroneous behavior): the property holds
  - If a counterexample is found: it may be spurious:
    It has to be verified with simulation and if it is spurious a new search has to be started

Structural properties

Properties of Petri nets independent from the initial state:

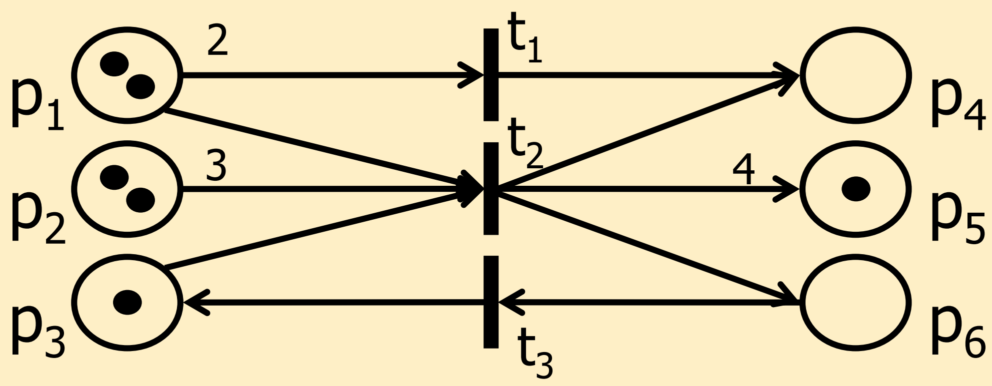
- Structural boundedness
- Controllability
- Conservativeness
  - Place invariant (P-invariant)
- Structural liveness
- Repetitiveness
- Consistence
  - Transition invariant (T-invariant)

Depending on the definition, the property must hold for

- either for all bounded initial marking,
- or some existing bounded initial marking

Recall: Describing the structure

- Weighted incidence matrix: **W** = [w(t, p)]
- Dimension: $\tau \times \pi = |T| \times |P|$
- w(t, p): Change in the number of tokens on p when t fires



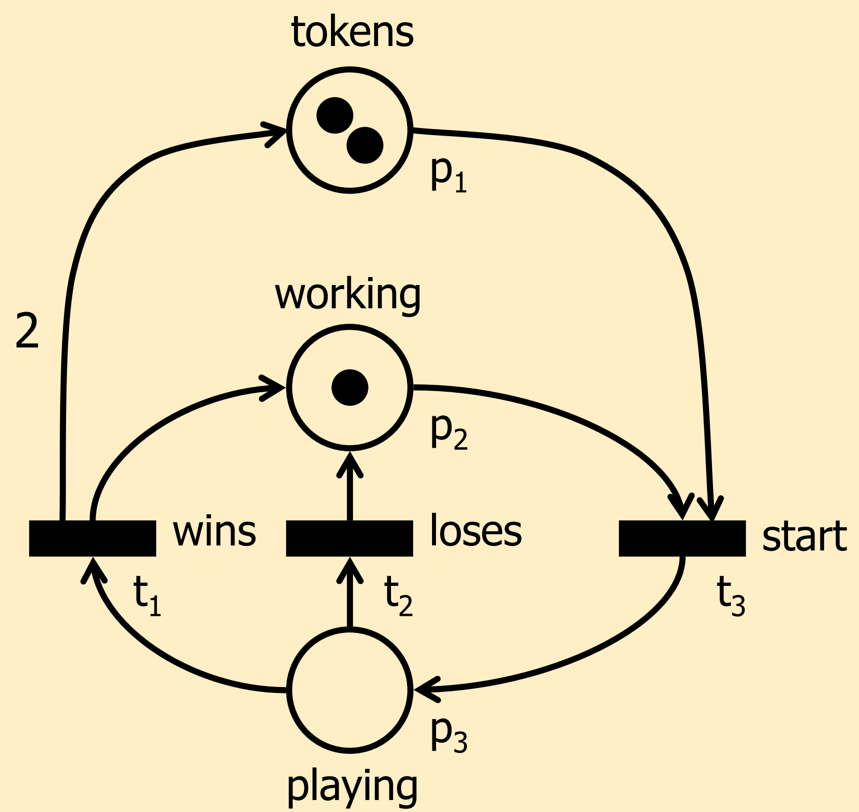
$\boldsymbol{W} = W^{+} - W^{-}$

$$W^{+} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$W^{-} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W = \begin{array}{c} \begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array} \\ \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ -1 & -3 & -1 & 1 & 4 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} \end{array}$$

Recall: Describing the structure

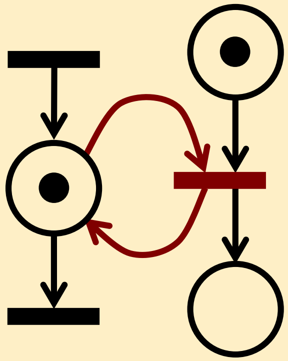


$$W = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \end{array}$$

$$W^T = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \end{array}$$

Introducing the state equation

- Dynamics of Petri nets: change in the marking
  - Changes can be described by equations
- Precondition (for unambiguousness): pure Petri net
  - No transition exists that is both the input and output transition of the same place: $\forall t \in T : \bullet t \cap t \bullet = \varnothing$
  - This subsumes: No “self-loop”
    - Marking does not change after firing (0 element in the incidence matrix)
    - But has a role in enabling the transition



Firing sequence

- Firing sequence:

$$\boxed{\vec{\sigma} = \left\langle M_{i_0} t_{i_1} M_{i_1} \ldots t_{i_n} M_{i_n} \right\rangle = \left\langle t_{i_1} \ldots t_{i_n} \right\rangle}$$

- Reachability of a state (marking):

$$M_{i_0} \left[\vec{\sigma} > M_{i_n} \right.$$

- Enabledness of a firing sequence:
  - Transition $t_{i,j}$ has enough tokens on input places $p \in \bullet t_{i,j}$

$$\boxed{\forall t_{i_j} \in \vec{\sigma}, \forall p \in \bullet t_{i_j} : M_{i_{j-1}}(p) \geq w^-(p, t_{i_j}) = \mathbf{W}^{-^{\mathrm{T}}} \vec{e}_{i_j}}$$

State equation

- Change in the marking:
  - When firing an enabled transition t_j
    - $w^{-}(p, t_j)$ tokens removed from each input place $p \in \bullet t_j$
    - $w^{+}(p, t_j)$ tokens are produced in each output place $p \in t_j\bullet$

$$\boxed{M_j = M_{j-1} - \mathbf{W}^{-^{\mathrm{T}}} \vec{e}_j + \mathbf{W}^{+^{\mathrm{T}}} \vec{e}_j = M_{j-1} + \mathbf{W}^{\mathrm{T}} \vec{e}_j}$$

  - When firing an enabled firing sequence $\underline{\sigma}$:
    - Marking changes by accumulating the firings:

$$\boxed{M_0 \left[\vec{\sigma} > M_j \quad \longrightarrow \quad M_j = M_0 + \mathbf{W}^{\mathrm{T}} \vec{\sigma}_T \right.}$$

- Firing count vector: number of occurrences for each transition in the firing sequence

Deriving the state equation

$$M_1 = M_0 + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_1}$$

$$M_2 = M_1 + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_2} = \overbrace{M_0 + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_1} + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_2}}^{\text{substituting } M_1}$$

$$\ldots$$

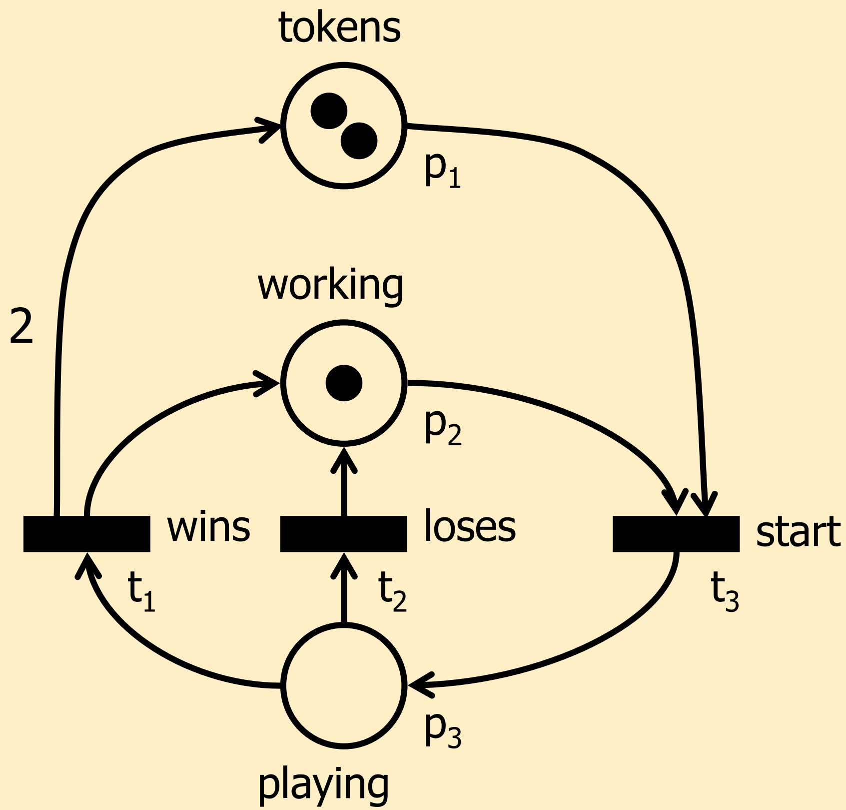
$$M_{n+1} = M_n + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_{n+1}} = M_0 + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_1} + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_2} + \ldots + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_{n+1}}$$

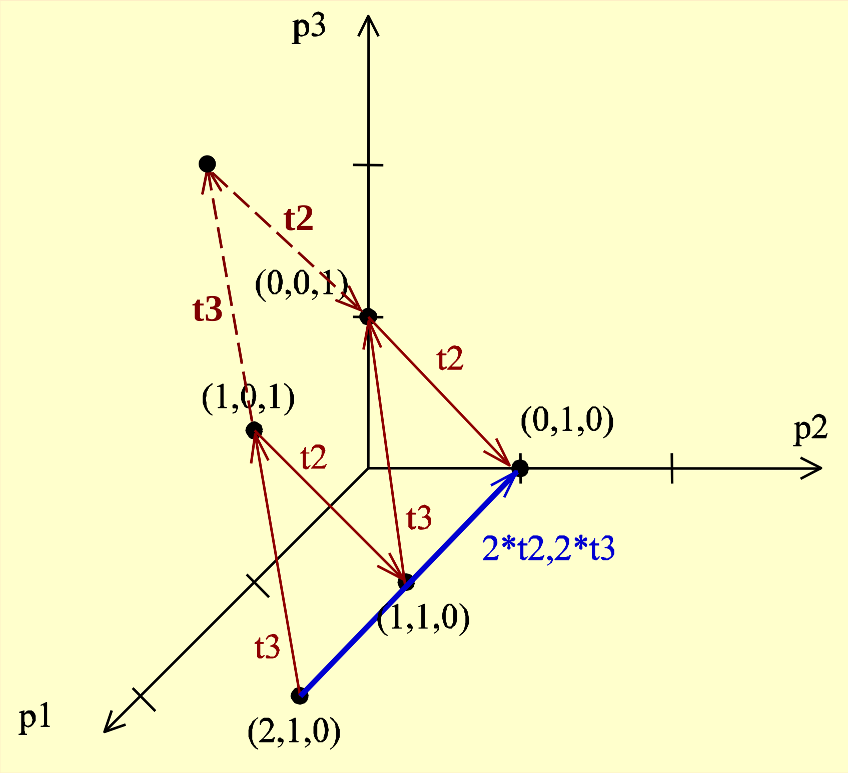
$$\ldots \qquad \Downarrow$$

$$M_m = M_0 + \underbrace{\mathbf{W}^{\mathrm{T}} \vec{e}_{t_1} + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_2} + \ldots + \mathbf{W}^{\mathrm{T}} \vec{e}_{t_m}}_{\text{joined}} = M_0 + \mathbf{W}^{\mathrm{T}} \sum_{i=1}^{m} \vec{e}_{t_i}$$

$$M_m = M_0 + \mathbf{W}^{\mathrm{T}} \vec{\sigma}_T \Rightarrow \boxed{M_m - M_0 = \mathbf{W}^{\mathrm{T}} \vec{\sigma}_T}$$

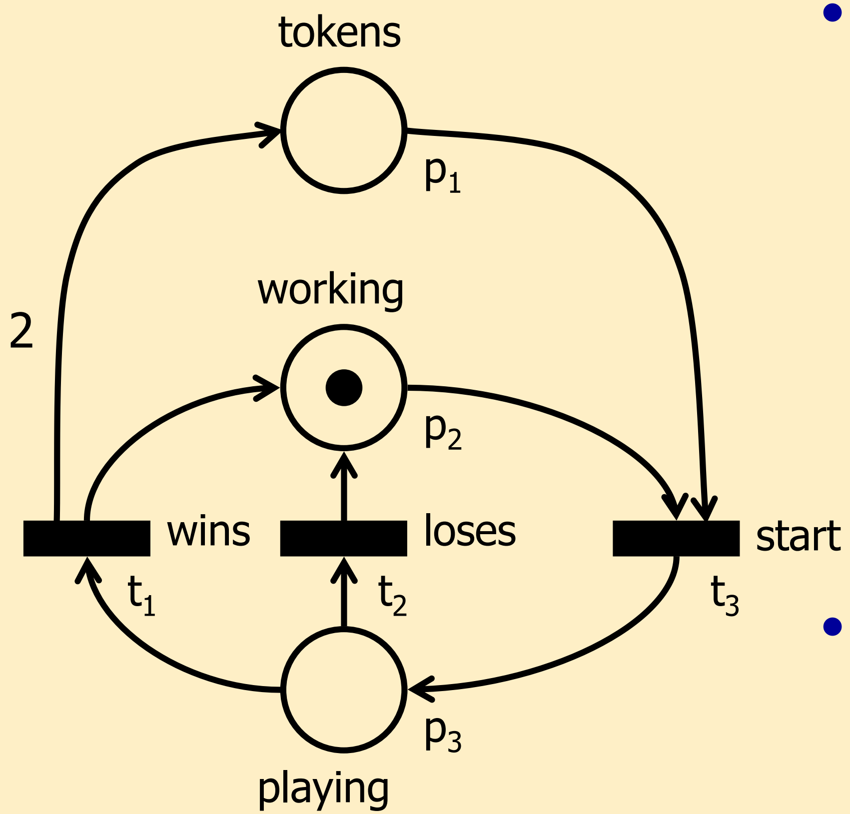
State equation and reachability





- The firing count vector contains less information, than the firing sequence
  - The order of firing is lost by only giving $(0,2,2)^T$!
  - A non fireable sequence can be obtained from the state equation for a given M_0

Example: State equation and reachability



- State equation:

$$\boxed{M_0[\vec{\sigma} > M_j \Rightarrow M_j - M_0 = W^T \vec{\sigma}_T}$$

$$\mathbf{W}^T = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \end{array}$$

- Firing count vector can be calculated to reach $(1,1,0)^T$ from $(0,1,0)^T$:

$$\boxed{(1,1,0)^T - (0,1,0)^T = \mathbf{W}^T \cdot (1,0,1)^T}$$

- Firing count vector: $(1,0,1)^T$
- But neither t_1, nor t_3 is enabled under the initial marking (0,1,0)!

Transition and place invariants

Definition: Transition invariant (T-invariant)

The firing count vector σ_T is a T-invariant, if its firing does not change the marking:

$$\boxed{\mathbf{W}^{\mathrm{T}}\vec{\sigma}_T = 0}$$

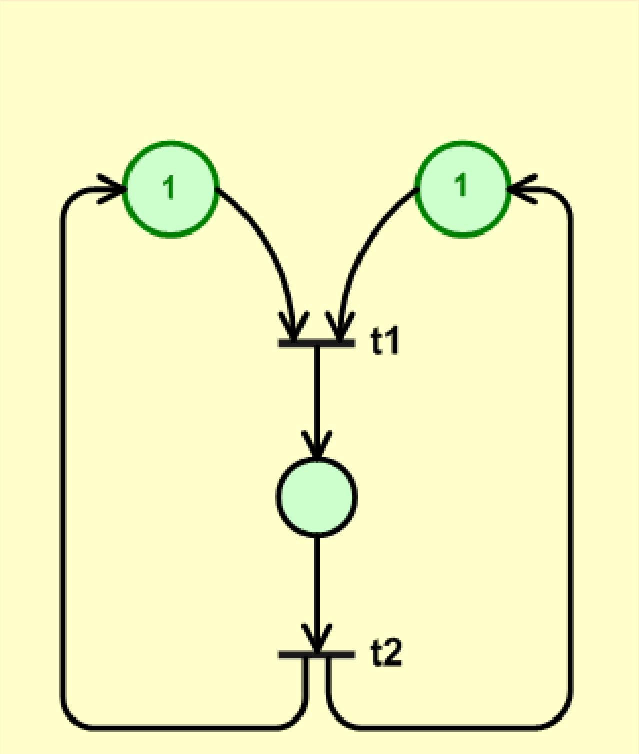
- Cycle in the state space: $M_i\left[\vec{\sigma}_T > M_i\right.$
- The firing sequence σ_T can be fired from state M_i if

$$\boxed{\forall t_{i_j} \in \vec{\sigma}, \forall p \in \left\{\bullet t_{i_j}\right\} : m_{i_{j-1}}\left(p\right) \geq w^{-}\left(p, t_{i_j}\right) = \mathbf{W}^{-\mathrm{T}} \cdot \vec{e}_{i_j}}$$

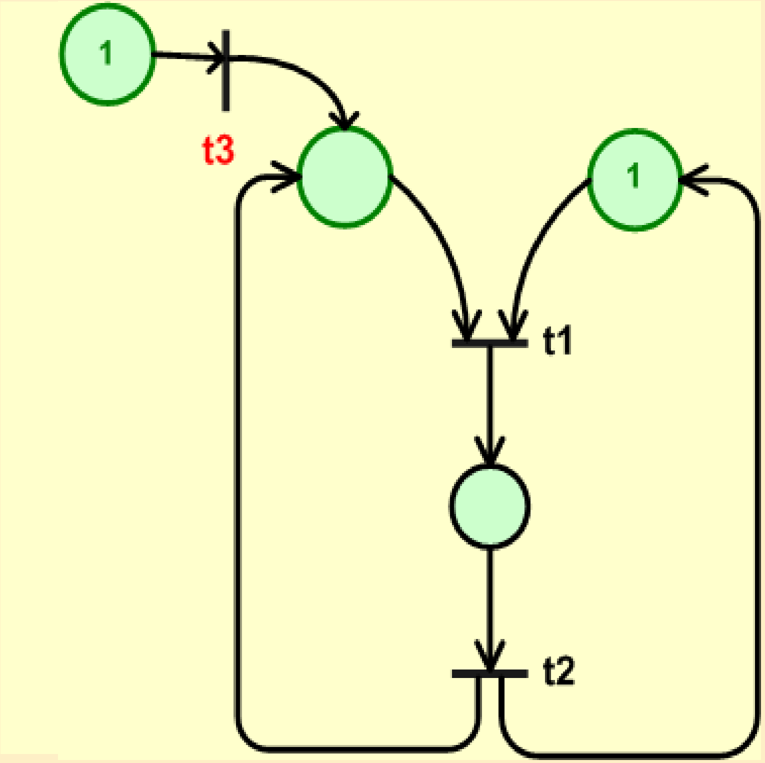
- Note: for each firing sequence σ an initial marking M_0 exists, from which σ can be fired
  - E.g. $M_0 \geq \mathbf{W}^{-\mathrm{T}}\vec{\sigma}$, the marking can have initially “as many” tokens, that the tokens produced by σ are not needed

Example T-invariant

T-invariant:
marking does not change after firing $t_1 - t_2$



Not a T-invariant:
firing sequence $t_3 - t_1 - t_2$ cannot be repeated



Set of T-invariants

$$\boxed{\mathbf{W}^{\mathrm{T}}\vec{\sigma}_{T} = 0}$$

Solutions of the homogeneous, linear system of equations

- Multiples of a solution are also solutions
  - If fireable, the loop can be traversed multiple times
- Sum of solutions is also a solution
  - If fireable, multiple loops can be combined
- Linear combination of solutions is also a solution

A basis can be found for the solutions

- Minimal set that can produce each solution

Minimal T-invariant

- Notation: basis of a firing sequence σ is $\sup(\sigma)$:
  - Set of transitions $T' = \{t_i \mid \sigma_i > 0\}$ occurring in the sequence σ
- T-invariant σ_T is minimal
  - If no T-invariant exists having a basis that is a proper subset of the basis of σ_T or
  - if the subsets are equal, its firing counts are lower

$$\boxed{\forall \sigma_T^1 : \mathbf{W}^{\mathrm{T}} \sigma_T^1 = 0 \Rightarrow \left(\sigma_T^1 \geq \sigma_T\right) \vee \left(\sup(\sigma_T) \not\subseteq \sup(\sigma_T^1)\right)}$$

Definition: Place invariant (P-invariant)

- A set of places marked by the non-negative weight vector μ_P, where the weighted sum of tokens is constant:

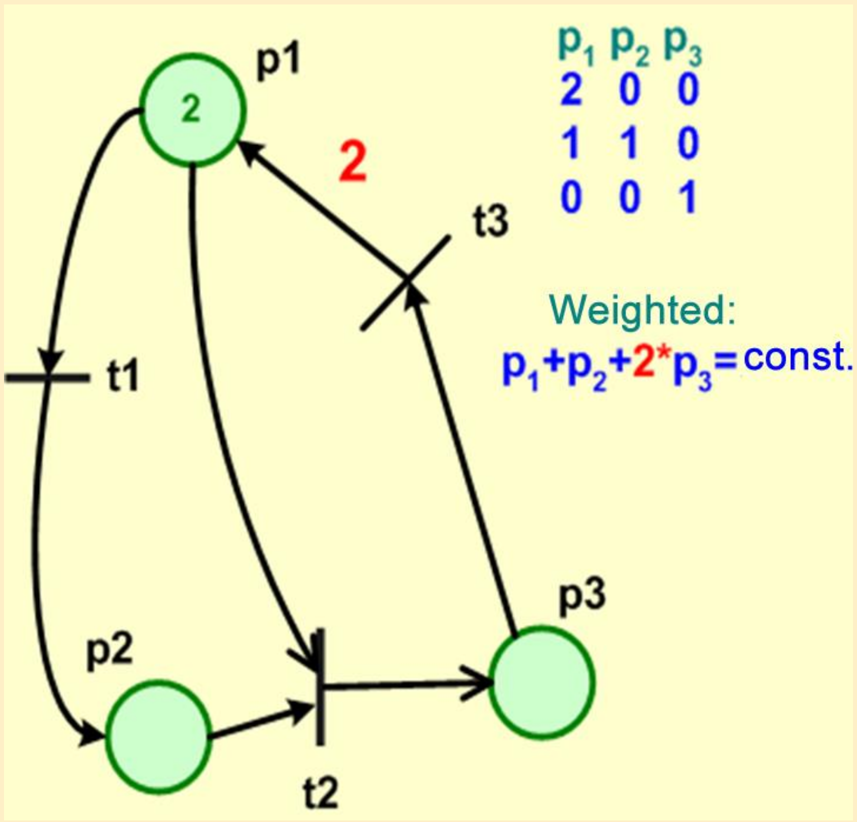
$$\boxed{\vec{\mu}_P^{\mathrm{T}} M = \text{constant}}$$

- Number of tokens in a subset of places is constant (e.g. resources are not lost or introduced)

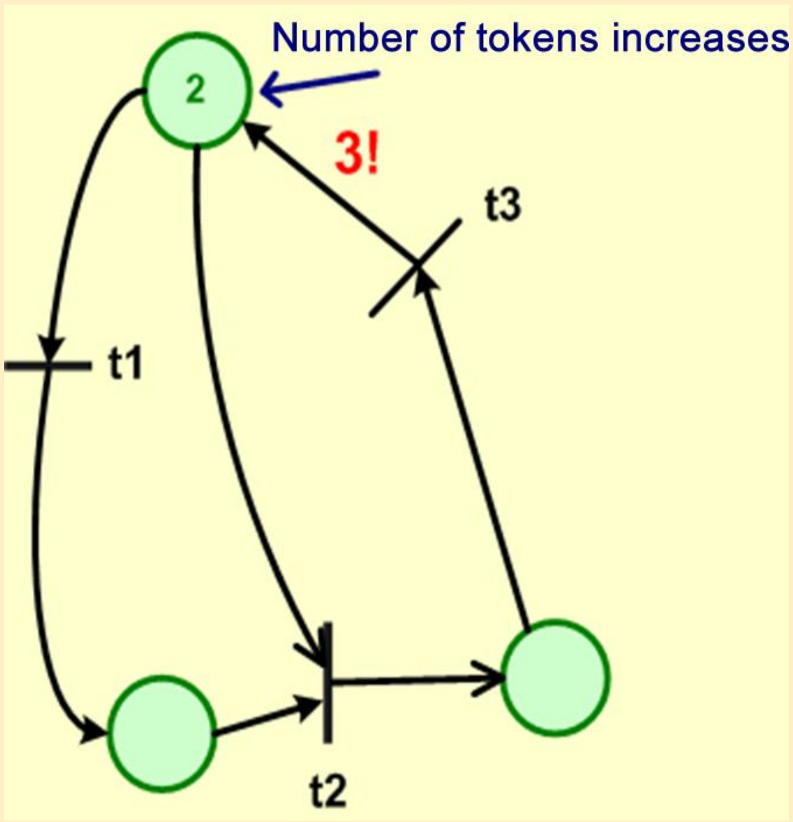
$$\left.\begin{aligned}
&M = M_0 + \mathbf{W}^{\mathrm{T}}\vec{\sigma} \\
&\underbrace{\vec{\mu}_P^{\mathrm{T}} M = \vec{\mu}_P^{\mathrm{T}} M_0}_{\vec{\mu}_P^{\mathrm{T}} M = \vec{\mu}_P^{\mathrm{T}} M_0 = \text{ constant}} + \vec{\mu}_P^{\mathrm{T}} \mathbf{W}^{\mathrm{T}} \vec{\sigma} \\
&\vec{\mu}_P^{\mathrm{T}} \mathbf{W}^{\mathrm{T}} \vec{\sigma} = 0 \Rightarrow \vec{\mu}_P^{\mathrm{T}} \mathbf{W}^{\mathrm{T}} \underset{\forall \vec{\sigma}}{\equiv} 0
\end{aligned}\right\} \quad \boxed{\mathbf{W}\vec{\mu}_P = 0}$$

Example P-invariant

P-invariant for p_1, p_2, p_3:



Not a P-invariant:

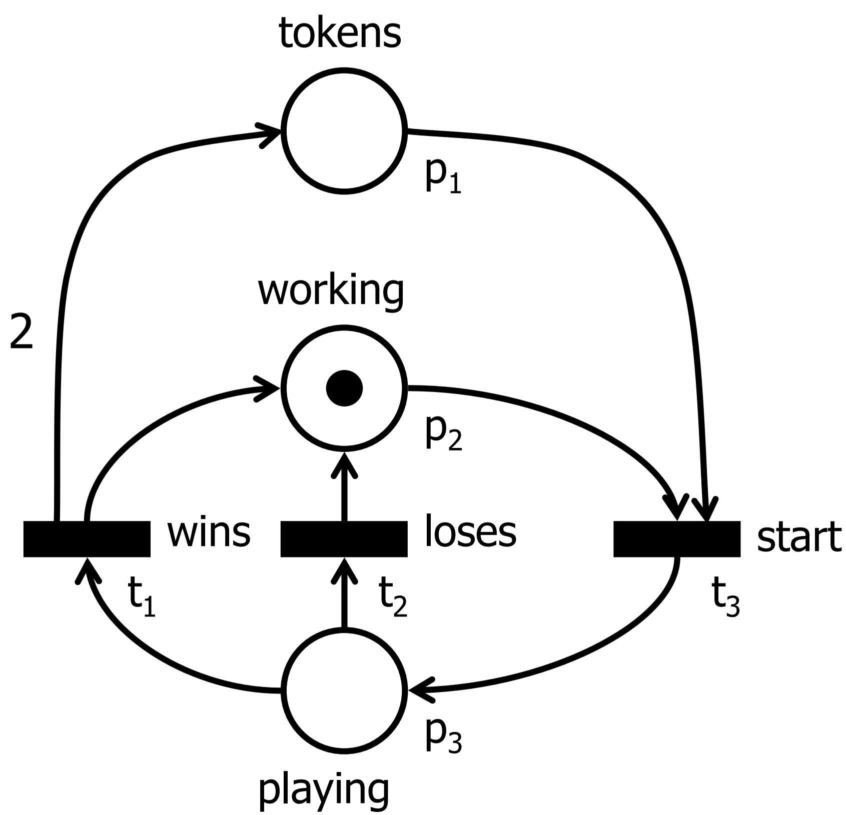


Applications of invariants

- Applications of T-invariants
  - For a process model: cyclical behavior
  - Dynamic properties
    - Cyclically fireable $\rightarrow$ reversibility, home state
    - Can be fired later $\rightarrow$ liveness, deadlock freedom
- Applications of P-invariants
  - For a process model: constant resources
  - Dynamic properties
    - Tokens are not lost $\rightarrow$ liveness, deadlock freedom
    - Tokens are not produced $\rightarrow$ boundedness

Calculating invariants

Does the example have invariants?



- For a P-invariant: $\mathbf{W} \cdot \mu_P = 0$

$$\mathbf{W} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \end{array}$$

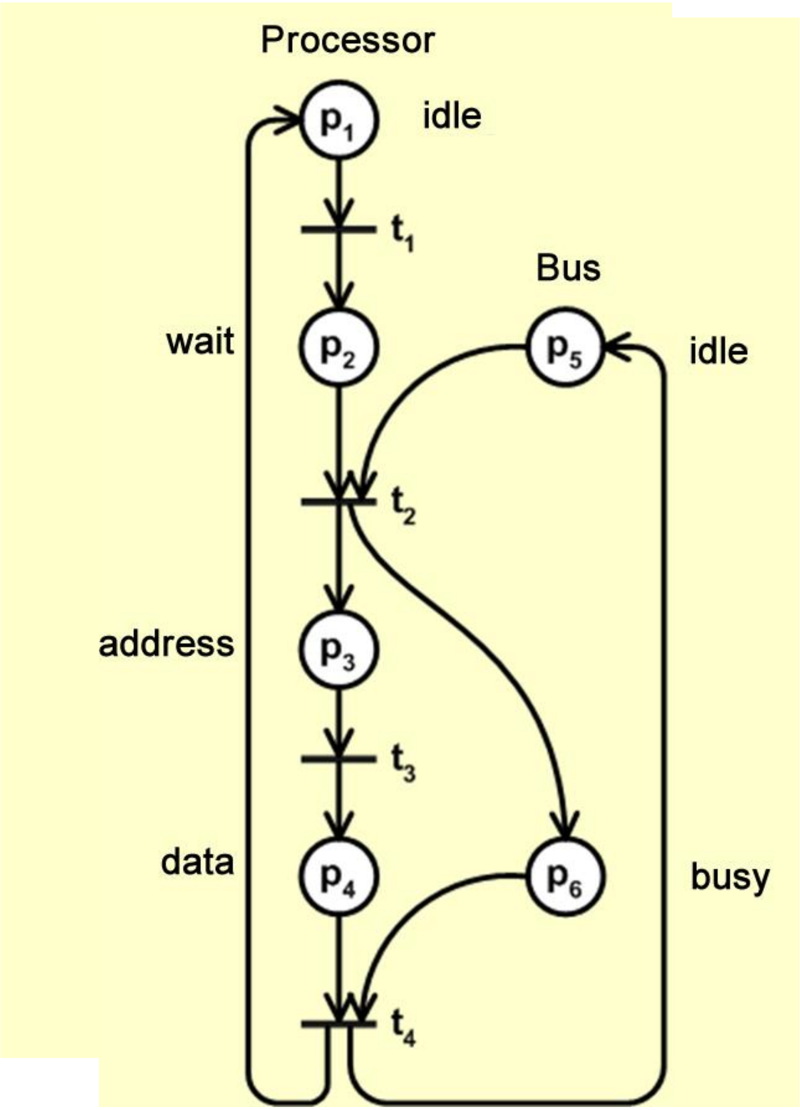
$$\mathbf{W} \cdot (0,\ \ 1,\ \ 1)^{\mathrm{T}} = 0$$

- For a T-invariant: $\mathbf{W}^{\mathrm{T}} \cdot \sigma_T = 0$

$$\mathbf{W}^{\mathrm{T}} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \end{array}$$

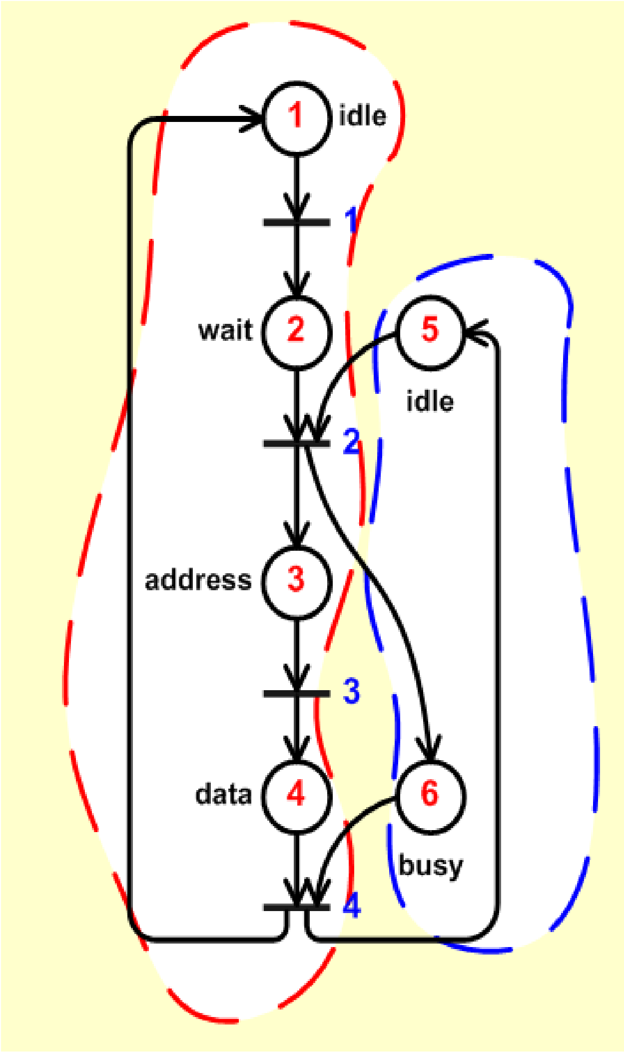
$$\mathbf{W}^{\mathrm{T}} \cdot (1,\ \ 1,\ \ 2)^{\mathrm{T}} = 0$$

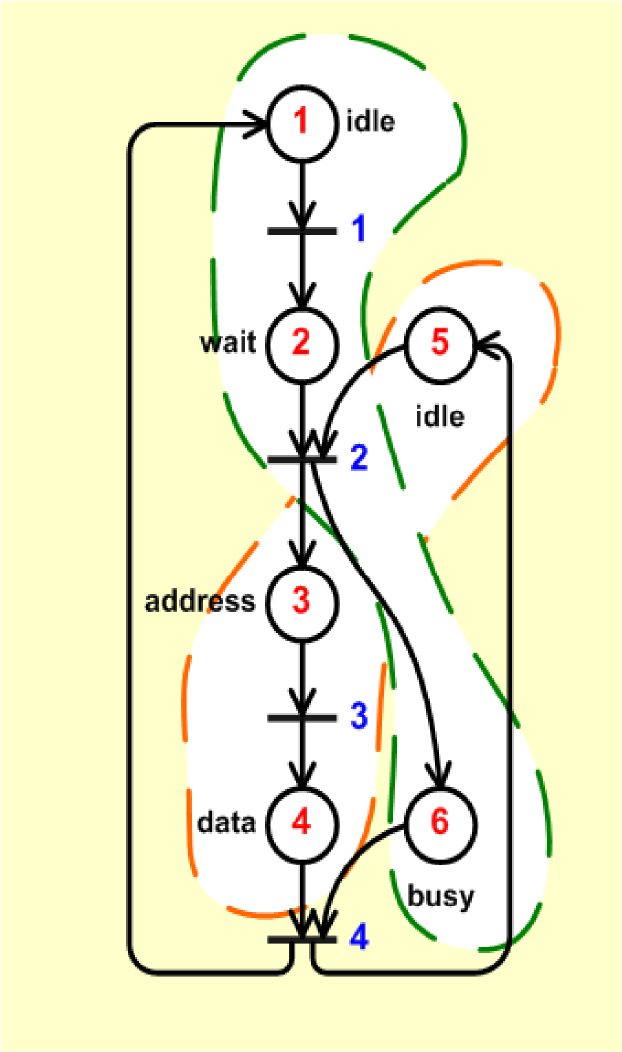
Example: Processor data transmission



- Processor
  - waiting (idle)
  - asking for bus grant
  - placing address to bus
  - placing data to bus
- Bus(es)
  - Idle (not used)
  - busy (processor/periphery)
- Petri net
  - n = 4 transitions
  - m = 6 places

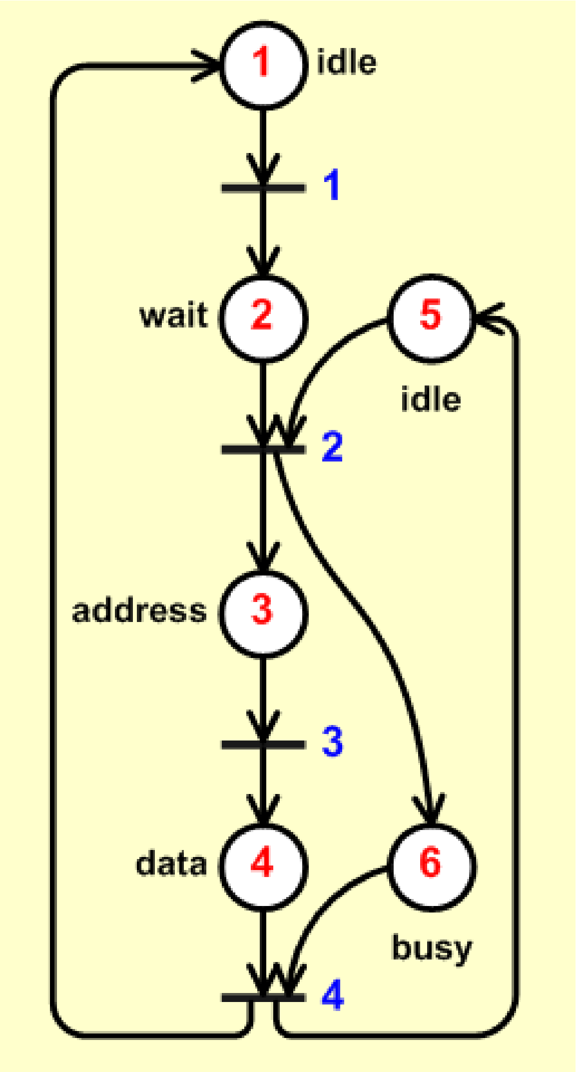
P-invariants: Calculate by hand!





Four P-invariants can be found

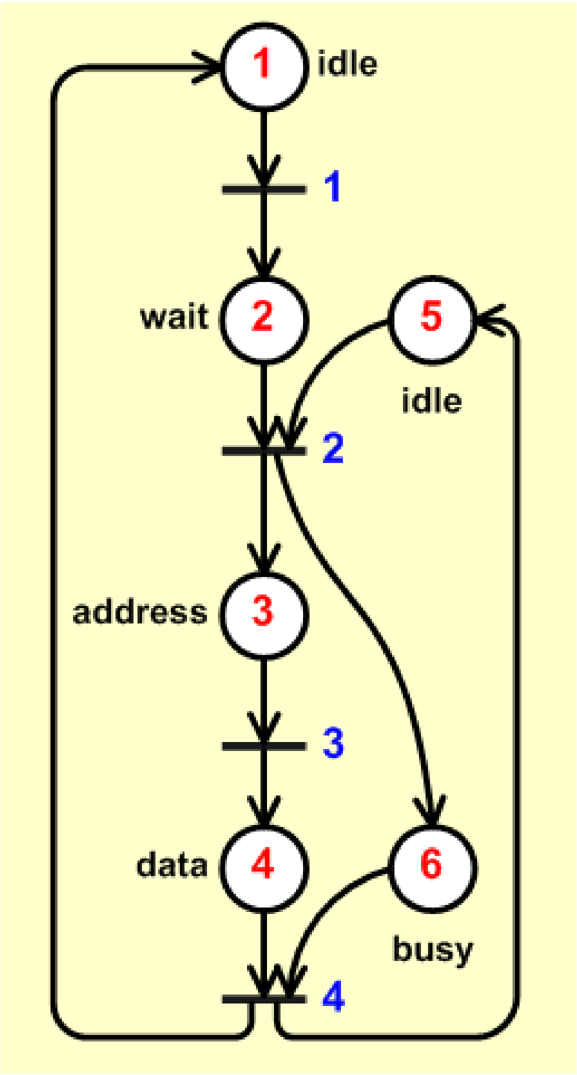
Example: Incidence matrices



$$W^- = \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{matrix}$$

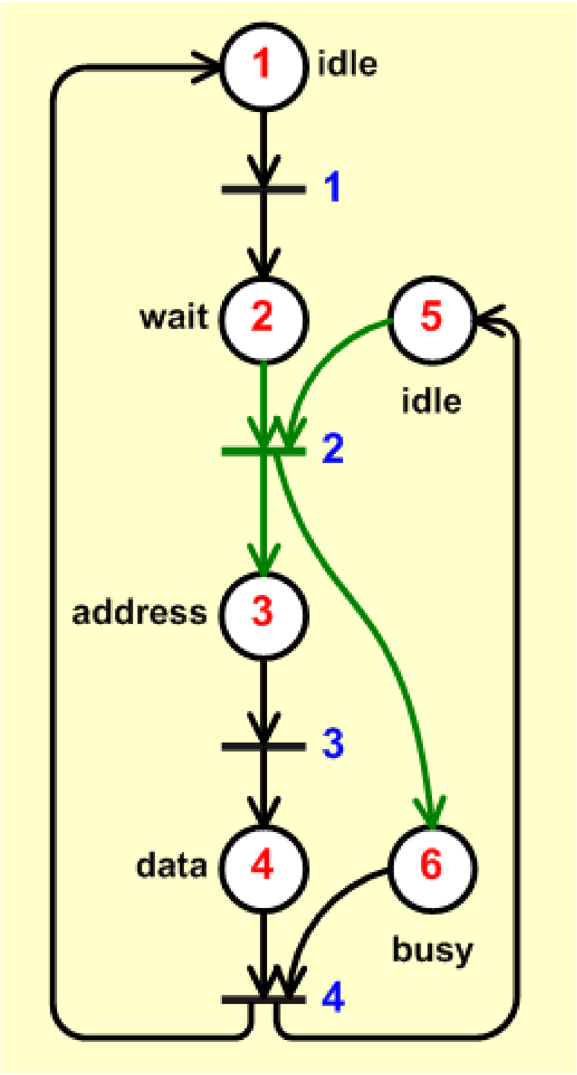
$$W^+ = \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{matrix}$$

Example: Incidence matrices



$$\mathbf{W} = W^{+} - W^{-} = \begin{array}{c} \begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array} \\ \left[\begin{array}{cccccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \end{array} \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}$$

Example: Incidence matrices



$$W^T = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc|c} t_1 & t_2 & t_3 & t_4 & \\ -1 & 0 & 0 & 1 & p_1 \\ 1 & -1 & 0 & 0 & p_2 \\ 0 & 1 & -1 & 0 & p_3 \\ 0 & 0 & 1 & -1 & p_4 \\ 0 & -1 & 0 & 1 & p_5 \\ 0 & 1 & 0 & -1 & p_6 \end{array}$$

Martinez-Silva algorithm: Initialization

$i \leftarrow 1$

$T_i \leftarrow \{ t \in T \}$

$\mathbf{A} \leftarrow \mathbf{W}^\mathrm{T}$, $\mathbf{D} \leftarrow \mathbf{1}_n$ // $n = |P|$

$\mathbf{Q}_i \leftarrow [\mathbf{D} \mid \mathbf{A}]$ // identity matrix and incidence matrix

$L_p \leftarrow$ the pth row of $\mathbf{Q}_i$

$T_1 = \{ t_1, t_2, t_3, t_4 \}$

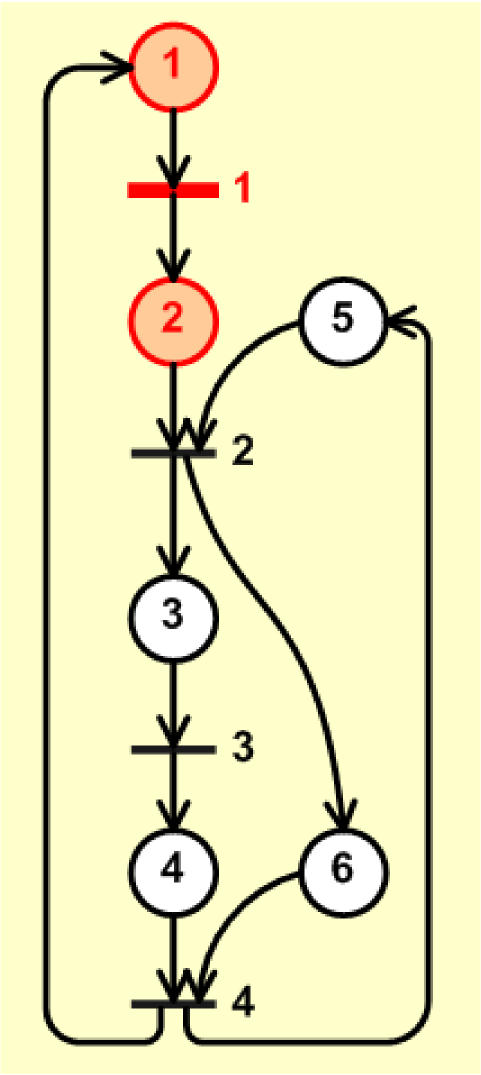
$$\mathbf{Q_1} = \begin{array}{c} \begin{array}{cccccc|cccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 \end{array} \\ \left[\begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right] \end{array} \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{array}$$

Martinez-Silva algorithm: Loop

```
while A_i ≠ 0
    if t_j ∈ T_i // choose a column not yet examined
    T_{i+1} ← T_i \ { t_j }
    L_delete ← ∅
    Q_{i+1} ← Q_i
    for all u, v : A_i(u, j) ≠ 0 ∧ A_i(v, j) ≠ 0 ∧
                   ∃λ_u, λ_v ∈ ∞+: λ_u A_i(u, j) + λ_v A_i(v, j) = 0
        add row λ_u L_u+λ_v L_v to Q_{i+1}
        L_delete ← L_delete ∪ { L_u, L_v }
    end for
    delete rows in L_delete from Q_{i+1}
    i ← i + 1
end while
```

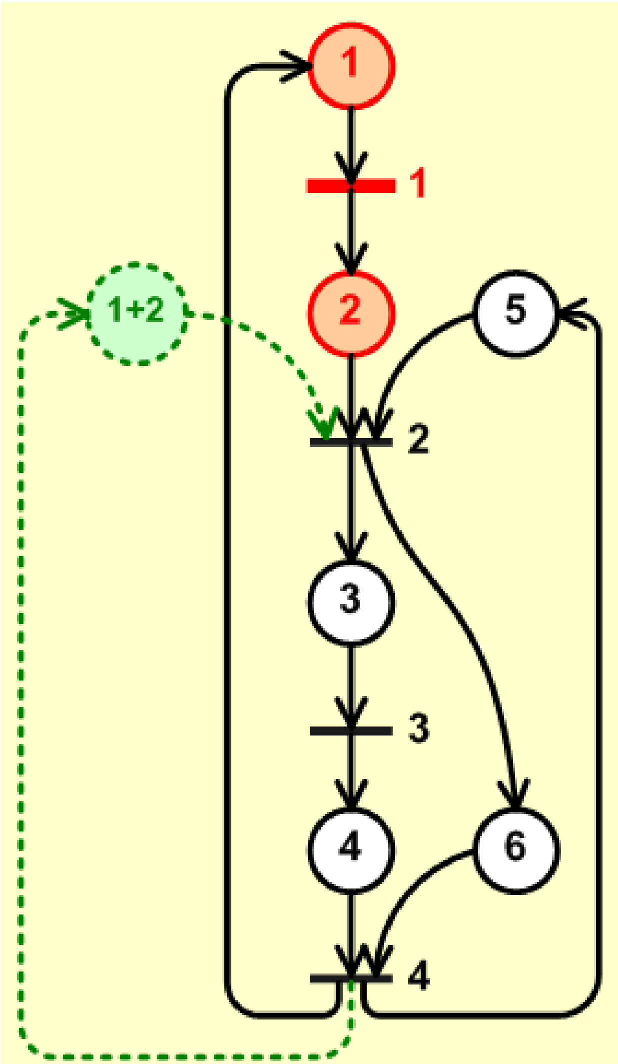
Find pairs of nonzero values in the jth column, whose weighted sum with given positive weights equals to 0

Martinez-Silva algorithm: Step 1/1



$$Q_1 = \begin{array}{cccccc|cccc|c}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 & \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & p_1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & p_2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & p_3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & p_4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & p_5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & p_6
\end{array}$$

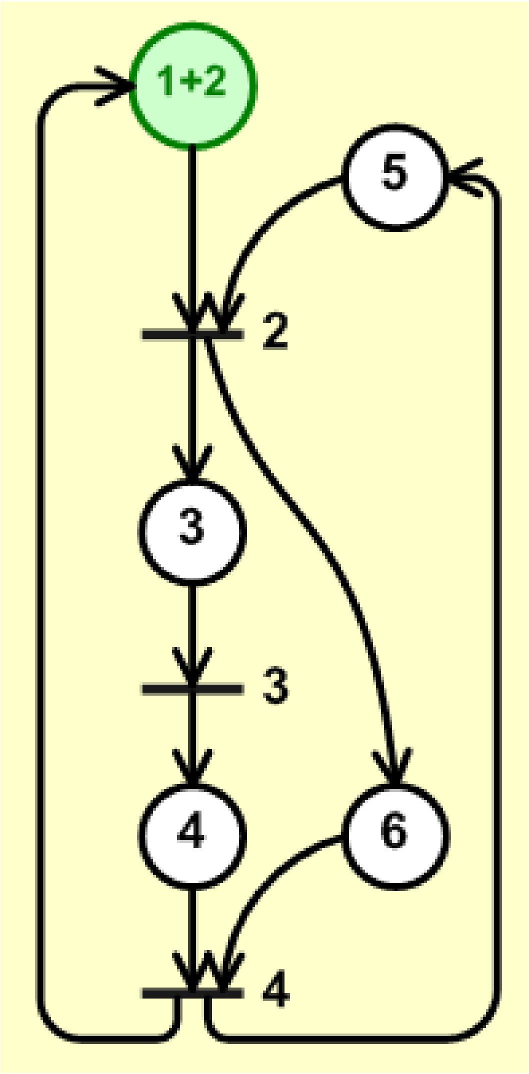
Martinez-Silva algorithm: Step 1/2



$$Q_1 = \begin{array}{cccccc|cccc|c} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 & \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & p_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & p_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & p_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & p_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & p_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & p_6 \end{array}$$

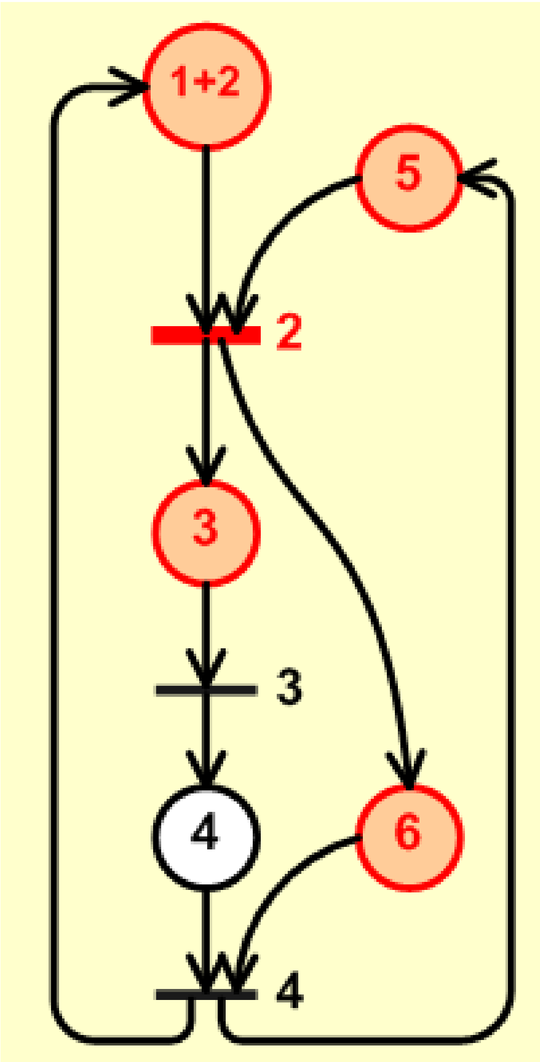
$$Q_1' = \begin{array}{cccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & p_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & p_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & p_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & p_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & p_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & p_6 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & p_{1+2} \end{array}$$

Martinez-Silva algorithm: Subresult 1



$$Q_1'' = \begin{array}{cccccc|cccc|l} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & p_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & p_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & p_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & p_6 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & p_{1+2} \end{array}$$

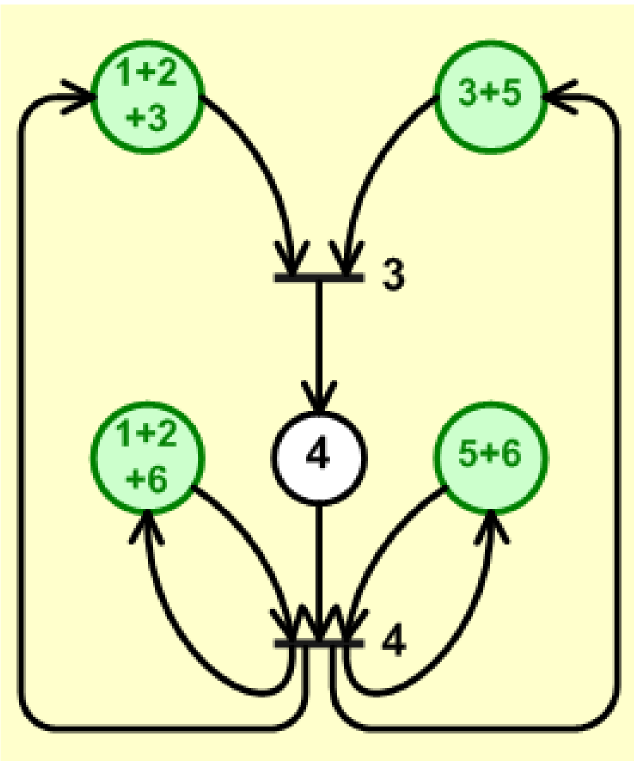
Martinez-Silva algorithm: Step 2/1, 2/2



$$Q_2 = \begin{array}{cl} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 \end{array} & \\ \left[\begin{array}{cccccc:cccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] & \begin{array}{l} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_{1+2} \end{array} \end{array}$$

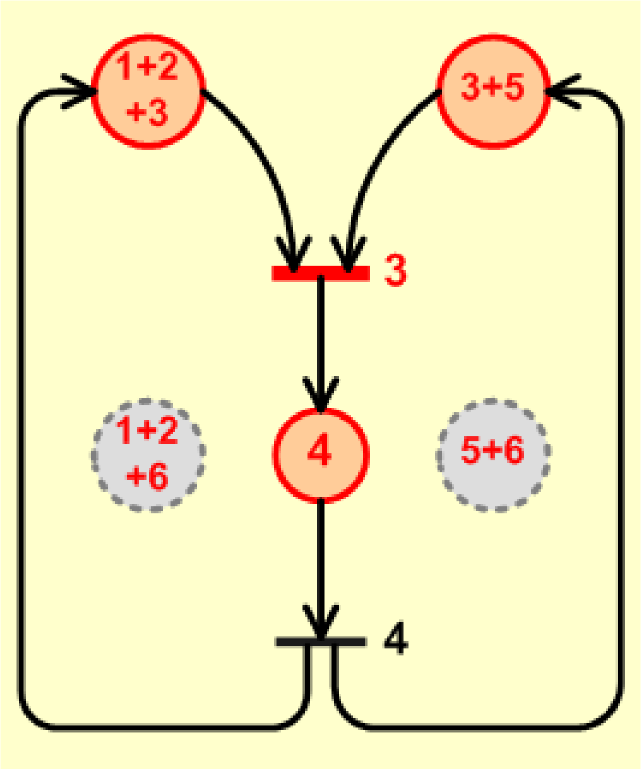
$$Q_2' = \left[\begin{array}{cccccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_{1+2} \\ p_{1+2+3} \\ p_{3+5} \\ p_{1+2+6} \\ p_{5+6} \end{array}$$

Martinez-Silva algorithm: Subresult 2



$$Q_2'' = \begin{array}{cl} \begin{array}{cccccc|cccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 \end{array} & \\ \left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{l} p_4 \\ p_{1+2+3} \\ p_{3+5} \\ p_{1+2+6} \\ p_{5+6} \end{array} \end{array}$$

Martinez-Silva algorithm: Step 3/1, 3/2

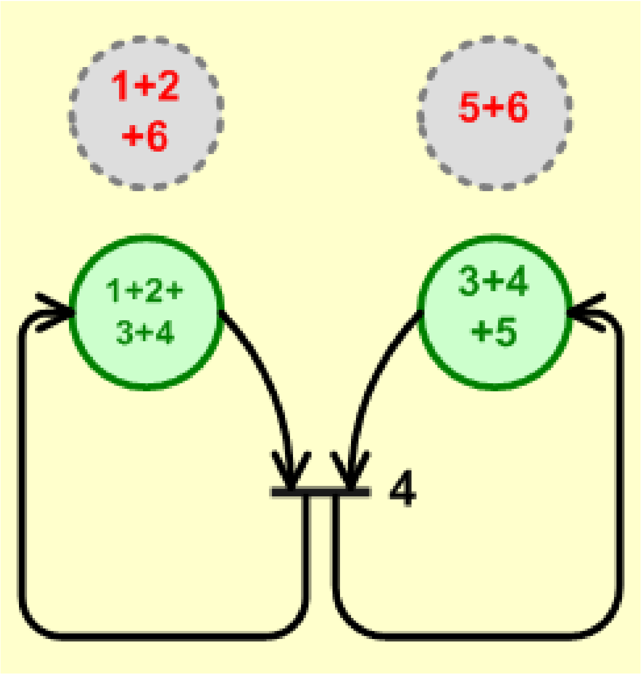


$$Q_3 = \begin{array}{c} \begin{array}{cccccc:cccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 \end{array} \\ \left[\begin{array}{cccccc:cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ p_4 \\ p_{1+2+3} \\ p_{3+5} \\ p_{1+2+6} \\ p_{5+6} \end{array}$$

$$Q_3' = \left[\begin{array}{cccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} p_4 \\ p_{1+2+3} \\ p_{3+5} \\ p_{1+2+6} \\ p_{5+6} \\ p_{1+2+3+4} \\ p_{3+4+5} \end{array}$$

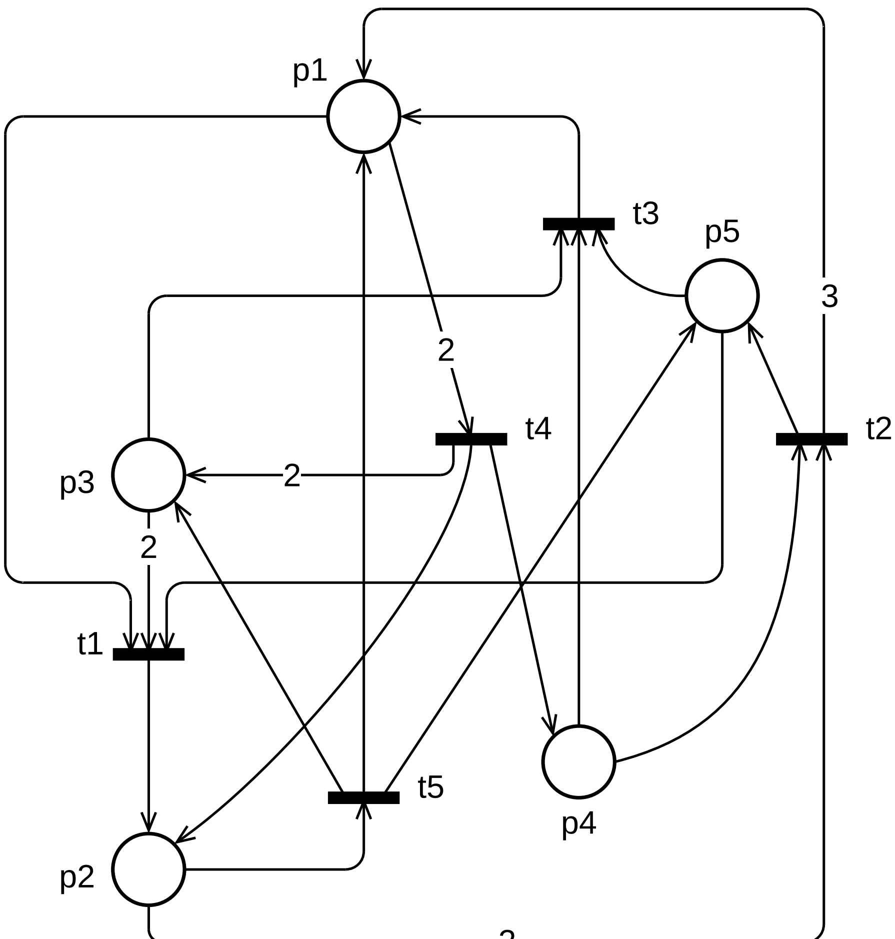
Martinez-Silva algorithm: Final results

$$Q_3'' = \begin{array}{cccccc|cccc|l}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & t_1 & t_2 & t_3 & t_4 & \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & p_{1+2+6} \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & p_{5+6} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & p_{1+2+3+4} \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & p_{3+4+5}
\end{array}$$



- Invariants:
  - Coefficients in the rows of matrix D_m in the final matrix $Q_m = [D_m|0]$
- Resulting P-invariants:
  1. $m(p_1)+m(p_2)+m(p_6) = 1$
  2. $m(p_5)+m(p_6) = 1$
  3. $m(p_1)+m(p_2)+m(p_3)+m(p_4) = 1$
  4. $m(p_3)+m(p_4)+m(p_5) = 1$
- Sum of tokens can be determined from the initial marking

Example: Calculating T-invariants



Start

						p_1	p_2	p_3	p_4	p_5		
t_1	1	0	0	0	0	-1	1	-2	0	-1	s_{11}	X (delete)
t_2	0	1	0	0	0	3	-2	0	-1	1	s_{12}	X
t_3	0	0	1	0	0	1	0	-1	-1	-1	s_{13}	X
t_4	0	0	0	1	0	-2	1	2	1	0	s_{14}	
t_5	0	0	0	0	1	1	-1	1	0	1	s_{15}	X

Step 1 (work with 5th column)

1	1	0	0	0	2	-1	-2	-1	0	(11+12)
0	1	1	0	0	4	-2	-1	-2	0	(12+13)
1	0	0	0	1	0	0	-1	0	0	(11+15)
0	0	1	0	1	2	-1	0	-1	0	(13+15)

(delete and reorder)

Before step 2

0	0	0	1	0	-2	1	2	1	0	s_{21}	X
1	1	0	0	0	2	-1	-2	-1	0	s_{22}	X
0	1	1	0	0	4	-2	-1	-2	0	s_{23}	X
1	0	0	0	1	0	0	-1	0	0	s_{24}	
0	0	1	0	1	2	-1	0	-1	0	s_{25}	X

Step 2 (work with 4th column)

1	1	0	1	0	0	0	0	0	0	(21+22)
0	0	1	1	1	0	0	2	0	0	(21+25)
0	1	1	2	0	0	0	3	0	0	(2*21+23)

(delete and reorder)

Before step 3

1	0	0	0	1	0	0	-1	0	0	s_{31}	X
1	1	0	1	0	0	0	0	0	0	s_{32}	
0	0	1	1	1	0	0	2	0	0	s_{33}	X
0	1	1	2	0	0	0	3	0	0	s_{34}	X

Step 3 (work with 3rd column)

2	0	1	1	3	0	(2*31+33)
3	1	1	2	3		(3*31+34)

Structural properties of Petri nets

Structural liveness, structural boundedness

- A Petri net N is structurally live, if there exists a live initial marking M_0 for N
  - A Petri net is live, if it is L4-live, i.e., each transition $t \in T$ is L4-live
    - A transition is L4-live: can be fired at least once in some firing sequence from any reachable state
- A Petri net N is structurally bounded, if it is bounded for all bounded initial markings M_0

Controllability

- A Petri net N is completely controllable, if for all bounded initial marking M_0 any marking is reachable from any other marking, i.e.,

$$\boxed{\forall M_i, M_j : M_i, M_j \in R(N, M_0) \Rightarrow M_i \in R(N, M_j) \wedge M_j \in R(N, M_i)}$$

Conservativeness

- A Petri net N is conservative, if there exists a positive integer weight μ_p for every place $p \in P$ in every bounded M_0 and $M \in R(N, M_0)$ such that:

$$\boxed{M\vec{\mu} = M_0\vec{\mu} = \text{constant}}$$

  - Example: For each initial marking, each place in each reachable marking is part of a P-invariant
- Partially conservative, if the above only holds for some places.
  - Example: For each initial marking, some places in each reachable marking is part of a P-invariants

Repetitiveness

- A Petri net *N* is repetitive, if an initial marking M_0 and a firing sequence σ from M_0 exists, such that every transition $t \in T$ occurs infinitely often in σ.
  - Example: An initial marking exists with a returning firing sequence (loop) containing every transition
- Partially repetitive, if the above only holds for some transitions.
  - Example: An initial marking exists with a returning firing sequence (loop) containing some transitions

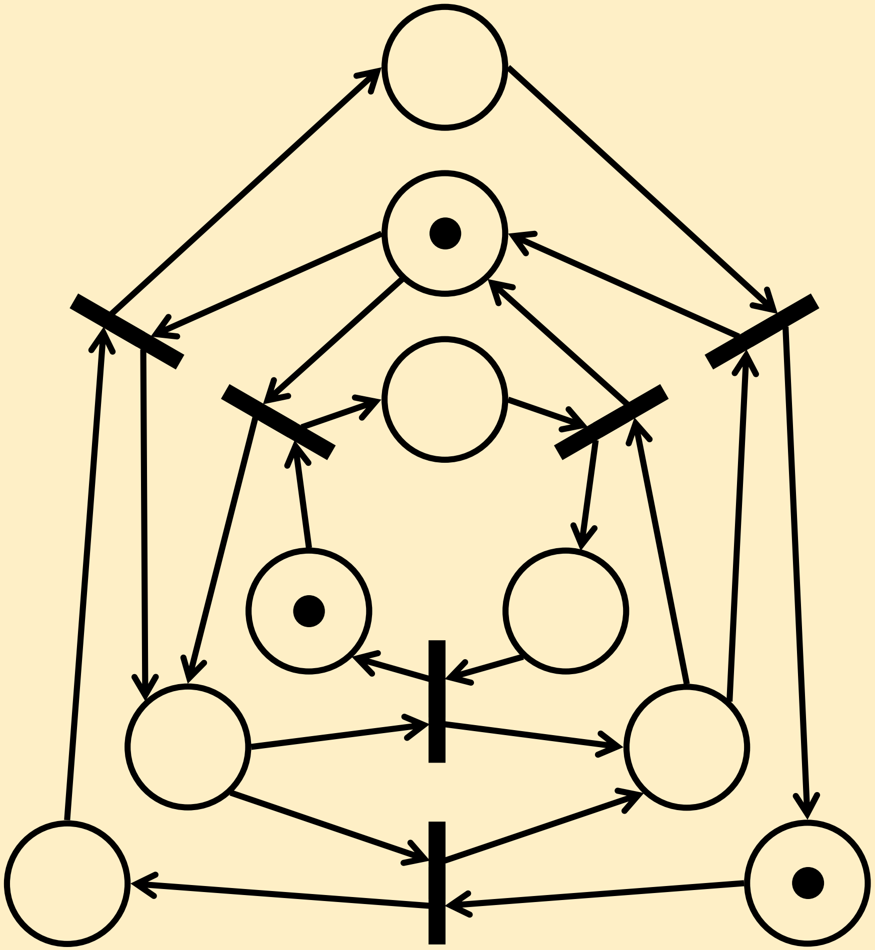
Consistency

- A Petri net N is consistent, if an initial marking M_0 and a firing sequence σ from M_0 to M_0 exists, such that every transition $t \in T$ occurs at least once in σ.
- Partially consistent, if the above only holds for some transitions.

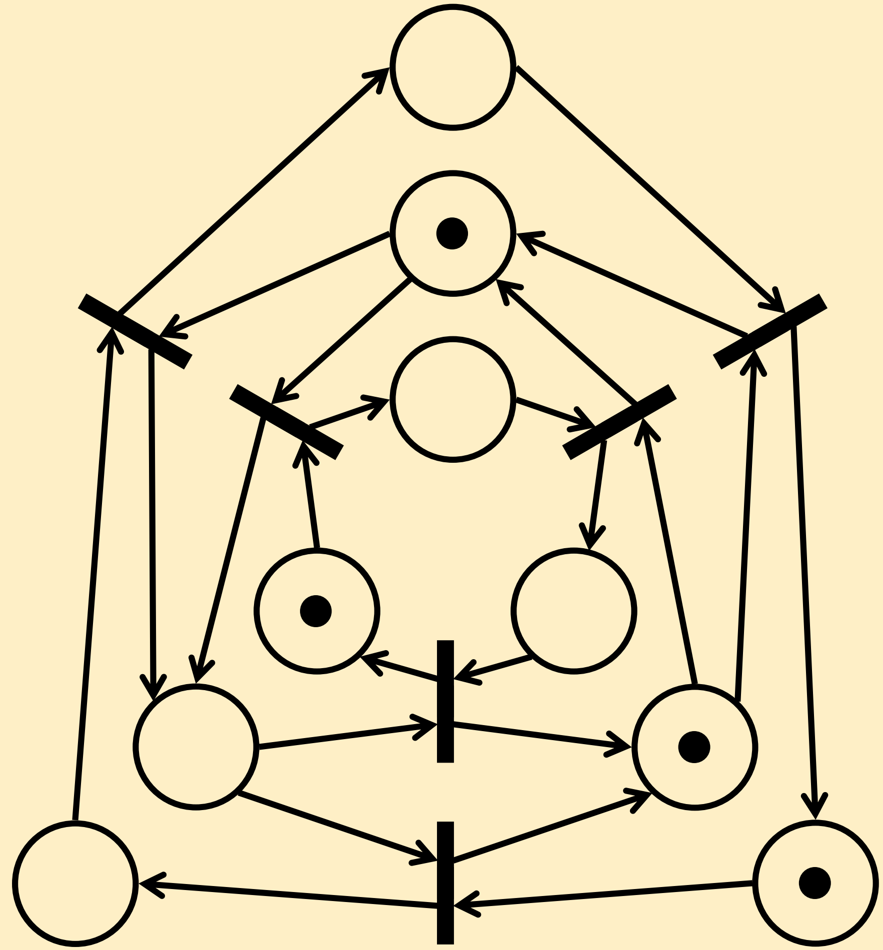
Structural B-fairness

- Two transitions are structurally B-fair, if for all initial markings M_0 the two transitions are B-fair
  - Two transitions are B-fair: One of them can fire only a bounded number of times without firing the other
- A Petri net *N* is structurally B-fair, if for all initial markings M_0 the net is B-fair
  - A Petri net (N, M_0) is B-fair, if any two transitions are in a B-fair relationship
  - Structural B-fair relation $\overset{\Longrightarrow}{\not\Longleftarrow}$ B-fair relation

B-fair, but not structurally B-fair net



B-fair M_0



Not B-fair M_0

Conditions for the properties*

	Property	Necessary and sufficient condition
SB	Structurally bounded	$\exists \vec{\mu} > 0, \mathbf{W}\vec{\mu} \le 0$ (or $\nexists \vec{\sigma} > 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} \underset{\neq}{\geq} 0$)
CN	Conservative	$\exists \vec{\mu} > 0, \mathbf{W}\vec{\mu} = 0$ (or $\nexists \vec{\sigma}, \mathbf{W}^{\mathrm{T}}\vec{\sigma} \underset{\neq}{\geq} 0$)
PCN	Partially conservative	$\exists \vec{\mu} \underset{\neq}{\geq} 0, \mathbf{W}\vec{\mu} = 0$
RP	Repetitive	$\exists \vec{\sigma} > 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} \geq 0$
PRP	Partially repetitive	$\exists \vec{\sigma} \underset{\neq}{\geq} 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} \geq 0$
CS	Consistent	$\exists \vec{\sigma} > 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} = 0$ (or $\nexists \vec{\mu}, \mathbf{W}\vec{\mu} \underset{\neq}{\geq} 0$)
PCS	Partially consistent	$\exists \vec{\sigma} \underset{\neq}{\geq} 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} = 0$

Other properties*

If …	Then …
N structurally bounded and structurally live	*N* is conservative and consistent.
$\exists \vec{\mu} \geq 0, \mathbf{W}\vec{\mu} \underset{\neq}{\leq} 0$	A non-live M_0 exists for *N*. *N* is not consistent.
$\exists \vec{\mu} \geq 0, \mathbf{W}\vec{\mu} \underset{\neq}{\geq} 0$	(N, M_0) is not bounded with live M_0. *N* is not consistent.
$\exists \vec{\sigma} \geq 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} \underset{\neq}{\leq} 0$	A non-live M_0 exists for structurally bounded *N*. *N* is not consistent.
$\exists \vec{\sigma} \geq 0, \mathbf{W}^{\mathrm{T}}\vec{\sigma} \underset{\neq}{\geq} 0$	*N* is not structurally bounded. *N* not conservative.