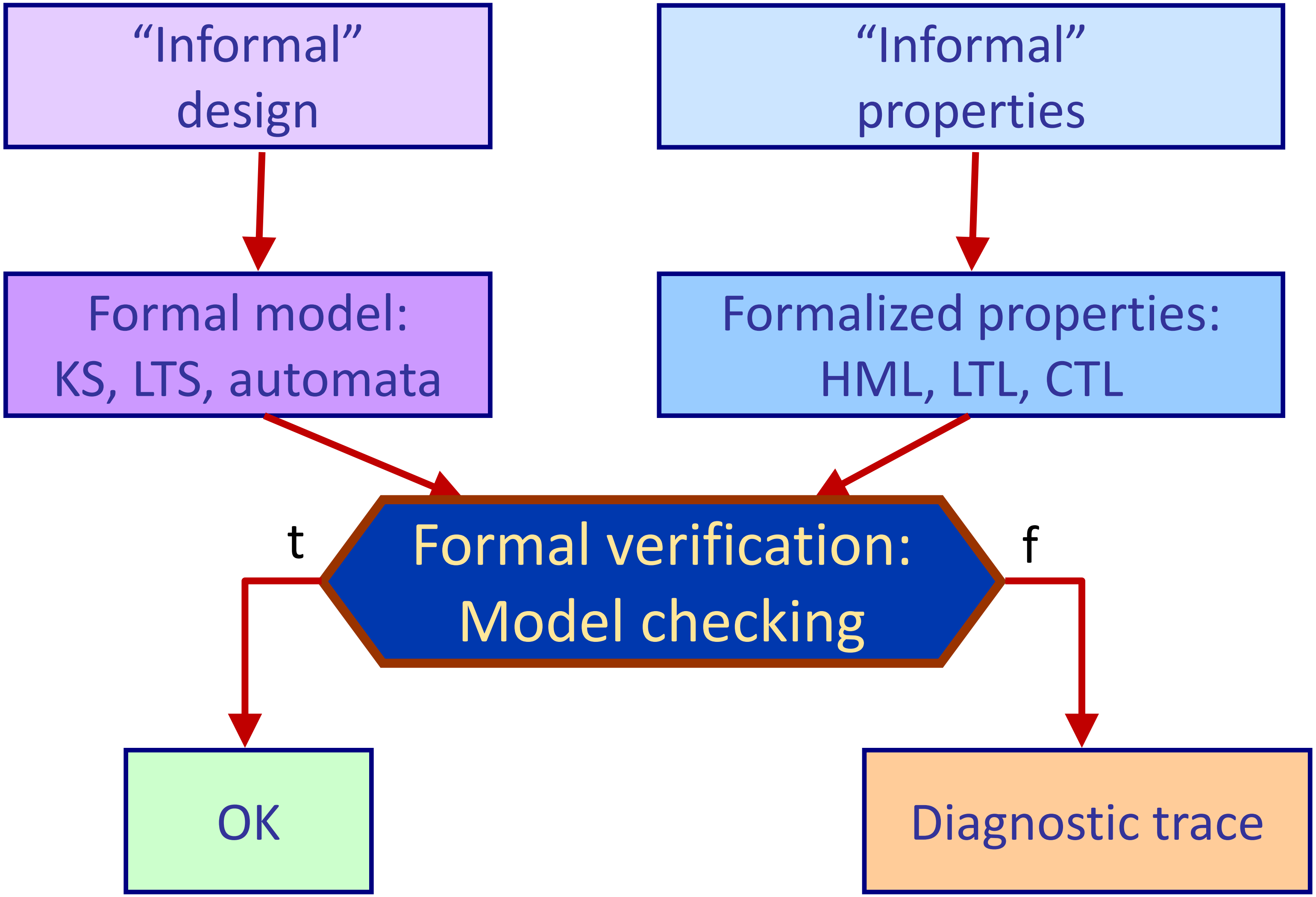
Model checking CTL: Symbolic technique

Istvan Majzik
majzik@mit.bme.hu

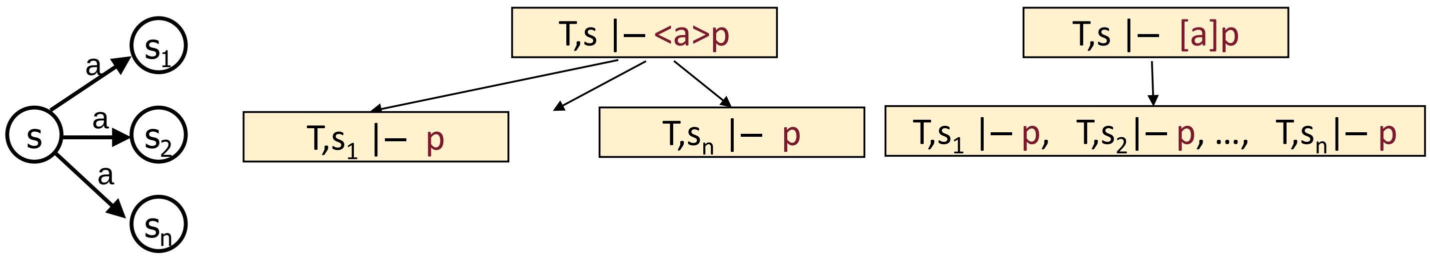
Budapest University of Technology and Economics
Dept. of Measurement and Information Systems

Formal verification of TL properties

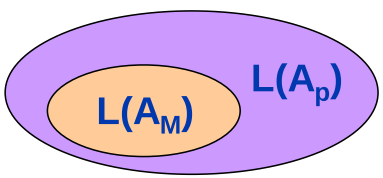


Recap: Techniques for model checking

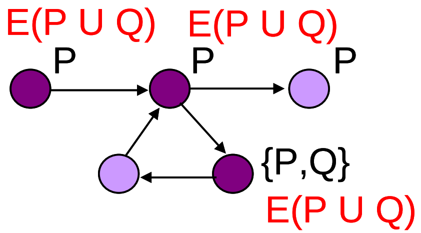
- HML model checking: Tableau-based



- LTL model checking: Based on automata-theory

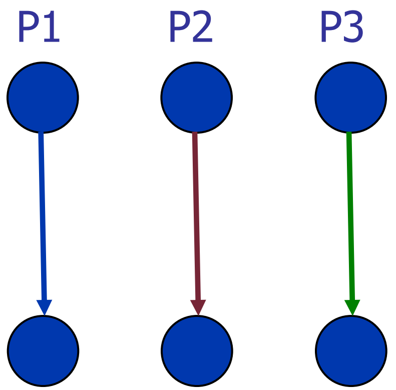


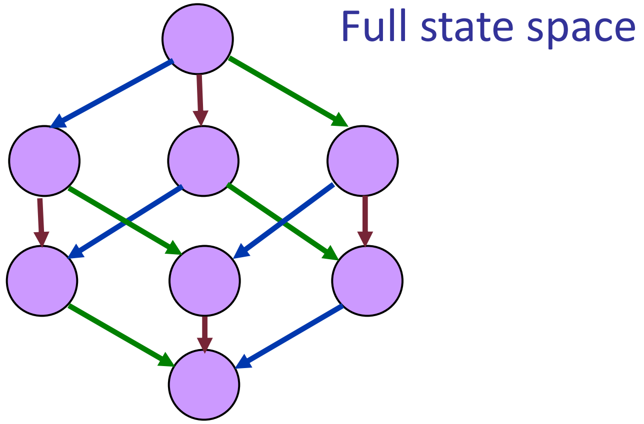
- CTL model checking: Iterative labeling



Problems

- The state space (e.g., Kripke structure) to traverse can be huge
  - Concurrent systems exhibit a large state space: Combinatorial explosion in the number of possible orderings of independent state transitions





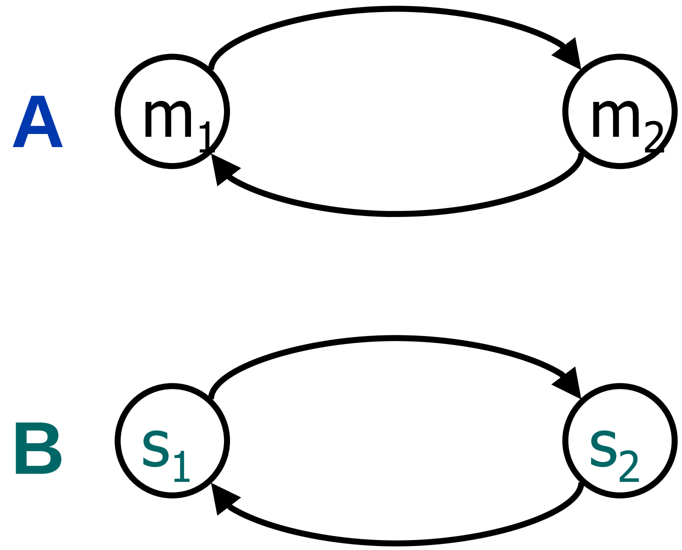
- How can we analyze large state spaces?
  - Promise: CTL model checking: 10^{20}, sometimes even 10^{100} states
  - What kind of technique can deliver this promise?

State space explosion

Direct product of automata, interleaving, synchronization

Operation of asynchronous automata

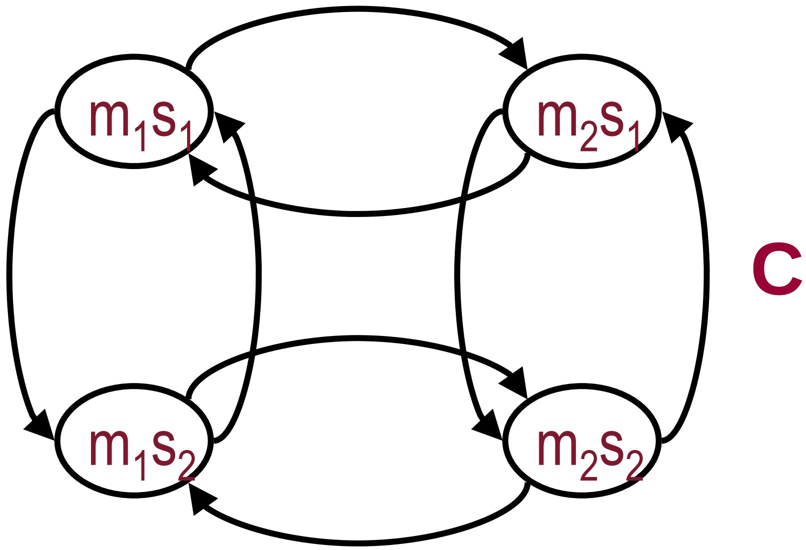
- System composed of two (independent) automata



- States of the automata:

  $A = \{m_1, m_2\}$, $B = \{s_1, s_2\}$

- (Direct) product automaton: state space of the system



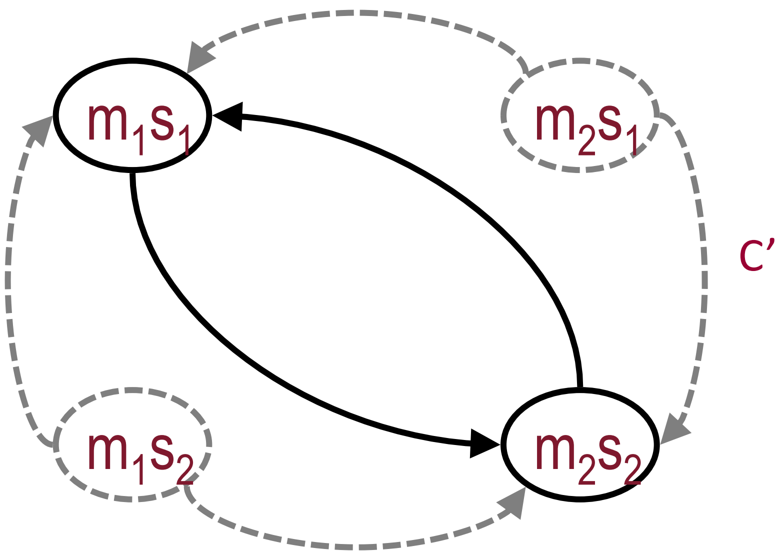
- Set of states:

  $C = A \times B$

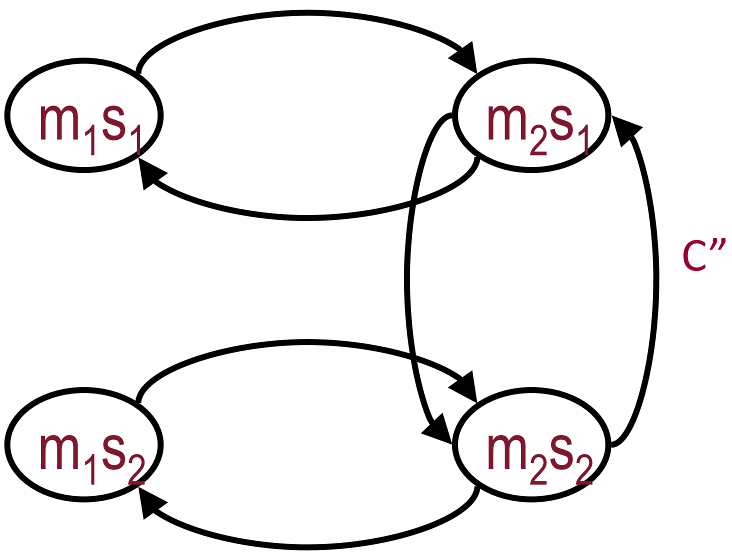
  $C = \{m_1s_1, m_1s_2 , m_2s_1 , m_2s_2\}$

The effects of synchronizations and guards

- Product automaton with synchronization: Taking the transitions at the same time
- E.g., A and B take their transitions simultaneously if their state index is the same:



- Product automaton with guards: Disable certain transitions
- E.g., B can only take the transition if A is in state m_2:

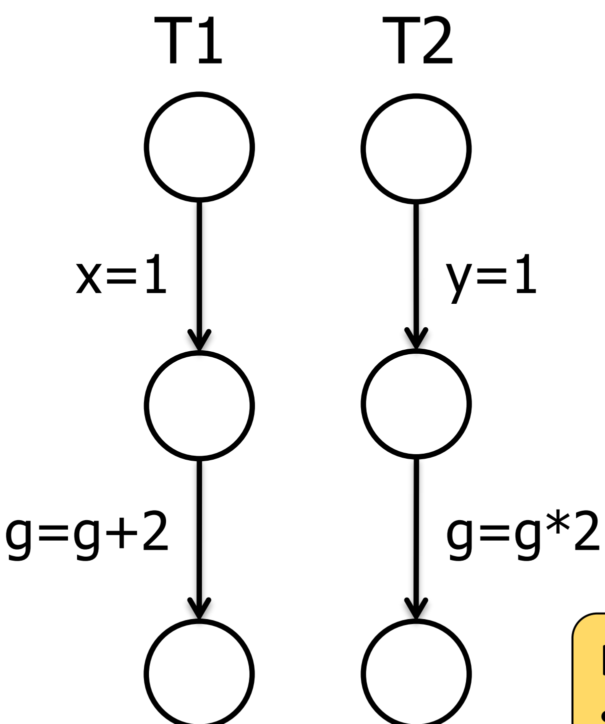


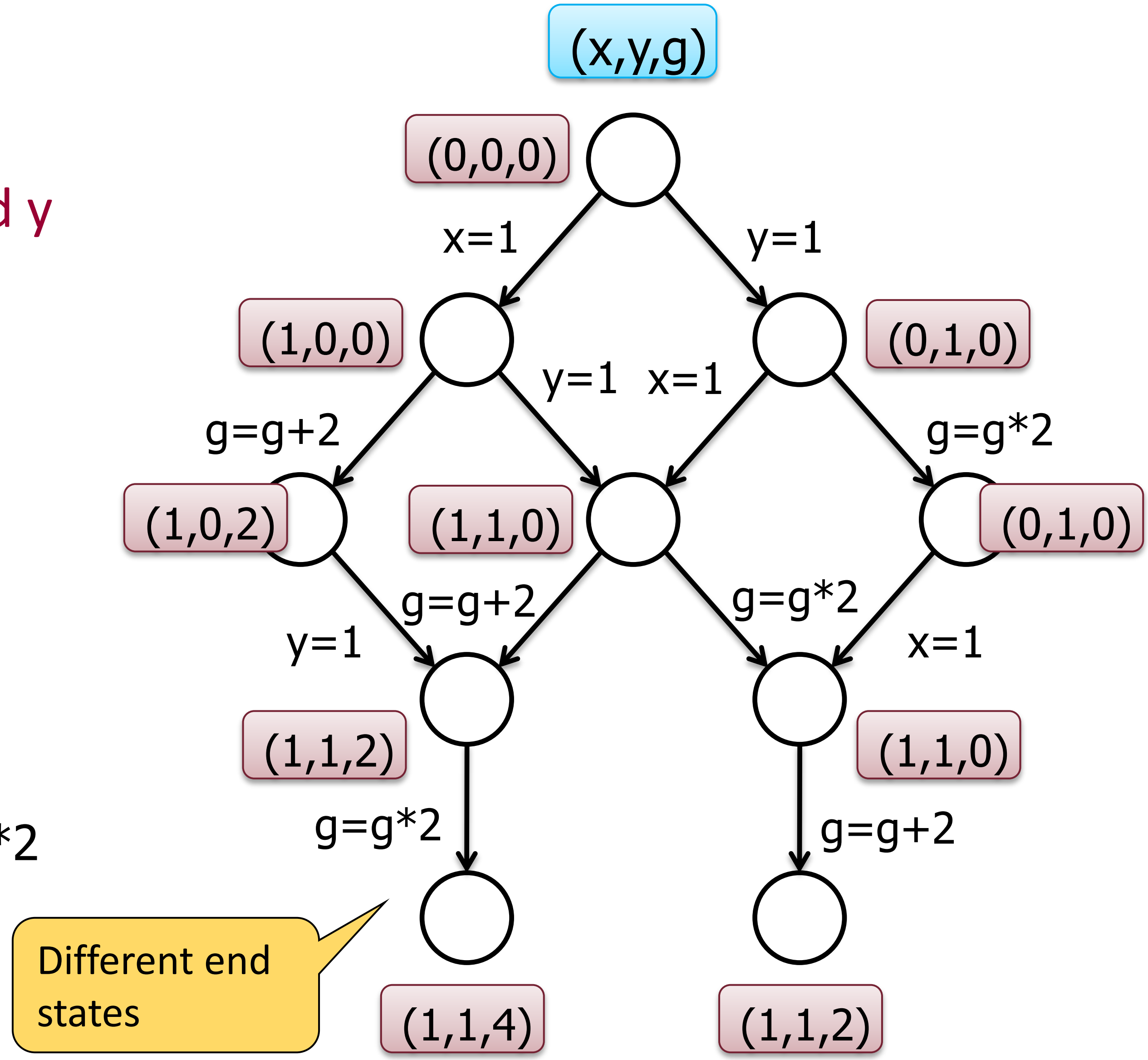
Data dependent states

Assigning actions to transitions

Local variables: x and y

Global variable: g





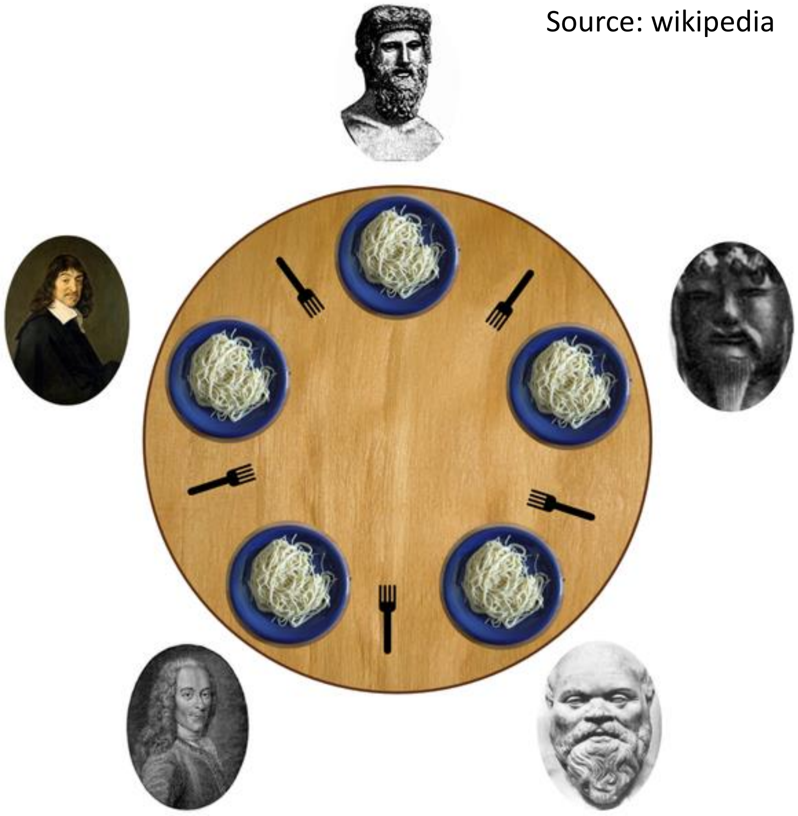
Example for large state space: Dining philosophers

- Concurrent system with non-trivial behavior
  - May have deadlock, livelock
- State space grows fast

#Philosophers	#States
16	$4{,}7 \cdot 10^{10}$
28	$4{,}8 \cdot 10^{18}$
…	…
200	$> 10^{40}$
1000	$> 10^{200}$
…	…

$2^{64} = 1{,}8 \cdot 10^{19}$

Source: wikipedia



With smart (but not task-specific) state space representation: ~100 000 philosophers, i.e. 10^{62900} states can be checked!

Techniques for handling large state space

- CTL model checking: Symbolic technique

Semantics-based technique	Symbolic technique
Sets of labeled states	Characteristic functions (Boolean functions): ROBDD representation
Operations on sets of states	Efficient operations on ROBDDs

- Model checking of invariants: Bounded model checking
  - Searching satisfying valuations for Boolean formulas with SAT techniques
  - Model checking to a given depth:
    Searching for counterexamples with bounded length
    - A detected counterexample is always valid
    - Non-existing counterexample does not imply correctness

Symbolic model checking

Recap: Model checking with set operations

- Set of states labeled with p is computed by set operations
  - E(p U q): “At least one successor is labeled …”
  - A(p U q): “All successors are labeled …”
- Notation: If the set of states labeled with p is Z then
  - **pre_E(Z)** = {s∈S | there exists s’, such that (s,s’)∈R and s’∈Z}
    i.e., at least one successor is labeled (is in Z)
  - **pre_A(Z)** = {s∈S | for all s’ where (s,s’)∈R: s’∈Z}
    i.e., all successors are labeled (are in Z)
- Example: Iterative labeling with E(P U Q)
  - Initial set: $X_0 = \{s \mid Q \in L(s)\}$
  - Next iteration: $X_{i+1} = X_i \cup (pre_E(X_i) \cap \{s \mid P \in L(s)\})$

States labeled so far, plus …

… their predecessor states that …

… are labeled with P

  - End of iteration: If $X_{i+1} = X_i$, the set is not increased

Main idea

- Representation of sets of states and operations on sets of states with Boolean functions
  - States are not explicitly enumerated
  - Encoding a state: with a bit-vector
    - To encode each state in S we need at least $n=\lceil \log_2|S| \rceil$ bits, so choose n such that $2^n \geq |S|$
  - Encoding a set of states: n-ary Boolean function called characteristic function
    - Characteristic function: $C: \{0,1\}^n \rightarrow \{0,1\}$
    - The characteristic function should be true for a bit-vector *iff* the state encoded by the bit-vector is in the given set of states
  - We will perform operations on characteristic functions instead of sets

Construction of characteristic functions

- For a state s: $C_s(x_1, x_2, \ldots, x_n)$

  Let the encoding of s be the bit-vector $(u_1, u_2, \ldots, u_n)$, where $u_i \in \{0,1\}$

  Goal: $C_s(x_1, x_2, \ldots, x_n)$ should return be true only for $(u_1, u_2, \ldots, u_n)$

  Construction of $C_s(x_1, x_2, \ldots, x_n)$: with operator $\wedge$:

  - x_i is an operand if $u_i=1$
  - $\neg x_i$ is an operand if $u_i=0$

  Example: for state s with encoding $(0,1)$: $C_s(x_1, x_2) = \neg x_1 \wedge x_2$

- For a set of states $Y \subseteq S$: $C_Y(x_1, x_2, \ldots, x_n)$

  Goal: $C_Y(x_1, x_2, \ldots, x_n)$ should be true for $(u_1, u_2, \ldots, u_n)$ iff $(u_1, u_2, \ldots, u_n) \in Y$

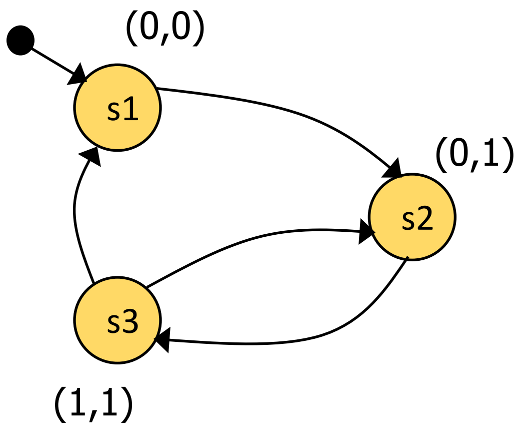
  Construction of $C_Y(x_1, x_2, \ldots, x_n)$ with operator $\vee$:

$$C_Y(x_1, x_2, \ldots, x_n) = \bigvee_{s \in Y} C_s(x_1, x_2, \ldots, x_n)$$

- For sets of states in general:

$$C_{Y \cup W} = C_Y \vee C_W, \qquad C_{Y \cap W} = C_Y \wedge C_W$$

Example: Characteristic function of states



Variables: x, y

Characteristic functions of states:

State s1:

$$C_{s1}(x,y) = (\neg x \wedge \neg y)$$

State s2:

$$C_{s2}(x,y) = (\neg x \wedge y)$$

State s3:

$$C_{s3}(x,y) = (x \wedge y)$$

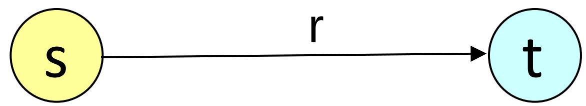
Characteristic function for a set of states:

Set of states {s1,s2}:

$$C_{\{s1,s2\}} = C_{s1} \vee C_{s2} = (\neg x \wedge \neg y) \vee (\neg x \wedge y)$$

Construction of characteristic functions (cont'd)

- For state transitions: C_r



$(u_1, u_2, \dots, u_n)$ $(v_1, v_2, \dots, v_n)$

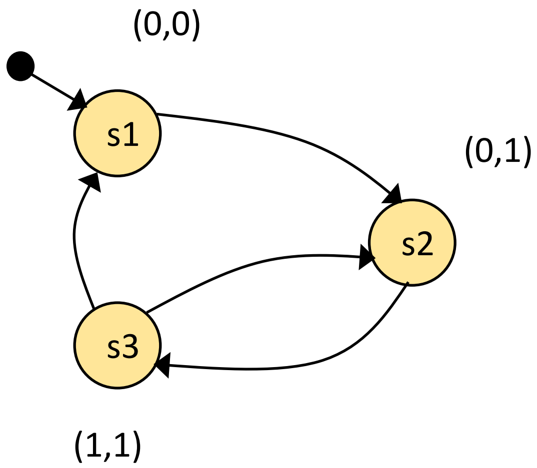
  - For transition $r=(s,t)$, where $s=(u_1, u_2, \dots, u_n)$ and $t=(v_1, v_2, \dots, v_n)$ characteristic function in the form $C_r(x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n)$
    - „Primed" variables denote the target state

  - Goal: C_r should be true *iff* $x_i=u_i$ and $x_i'=v_i$

    Construction of C_r:

$$C_r = C_s(x_1, x_2, \dots, x_n) \wedge C_t(x'_1, x'_2, \dots, x'_n)$$

Example: Characteristic functions of transitions



State s1 encoded by (0,0):

$$C_{s1}(x,y) = (\neg x \wedge \neg y)$$

State s2 encoded by (0,1):

$$C_{s2}(x,y) = (\neg x \wedge y)$$

Transition (s1,s2)$\in$R, i.e., (0,0) $\rightarrow$ (0,1):

$$C_{(s1,s2)} = (\neg x \wedge \neg y) \wedge (\neg x' \wedge y')$$

Transition relation R:

$$\begin{aligned} R(x,y,x',y') = & (\neg x \wedge \neg y \wedge \neg x' \wedge y') \vee \\ & \vee (\neg x \wedge y \wedge x' \wedge y') \vee \\ & \vee (x \wedge y \wedge \neg x' \wedge y') \vee \\ & \vee (x \wedge y \wedge \neg x' \wedge \neg y') \end{aligned}$$

Construction of characteristic functions (cont'd)

- Construction of $\text{pre}_E(Z)$: $\text{pre}_E(Z)=\{s \mid \exists t: (s,t)\in R \text{ and } t\in Z\}$

  Representation of Z: function C_Z

  Representation of R: function $C_R=\bigvee_{r\in R} C_r$

  $\text{pre}_E(Z)$: find predecessor states for states of Z

$$C_{\text{pre}_E(Z)} = \exists_{x'_1, x'_2, \ldots, x'_n} C_R \wedge C'_Z$$

  where $\exists_x C = C[1/x] \vee C[0/x]$ („existential abstraction")

- Model checking with set operations: reduced to operations with Boolean functions
  - Union of sets: Disjunction of functions ($\bigvee$)
  - Intersection of sets: Conjunction of functions ($\bigwedge$)
  - Construction of $\text{pre}_E(Z)$: Complex operation (existential abstraction)

Representation of Boolean functions

Canonic form: ROBDD

Reduced, Ordered Binary Decision Diagram

Conceptual construction of ROBDD (overview):

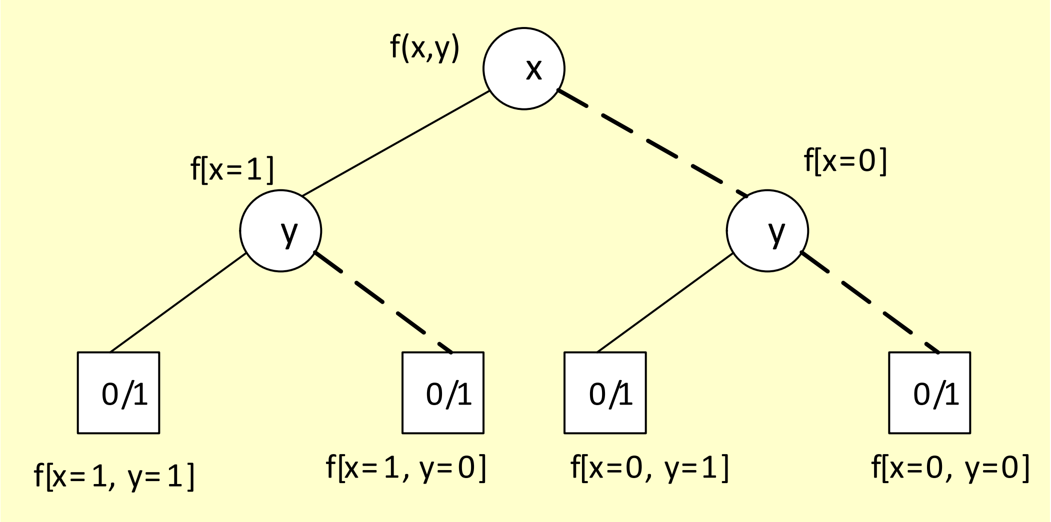
- Binary decision tree: Represents binary decisions given by the valuation of function variables
- BDD: Identical subtrees are merged
- OBDD: Evaluation of variables in the same order on every branch
- ROBDD: Reduction of redundant nodes
  - If both two outcomes (branches) lead to the same node

ROBDD in more detail

Types of decision trees

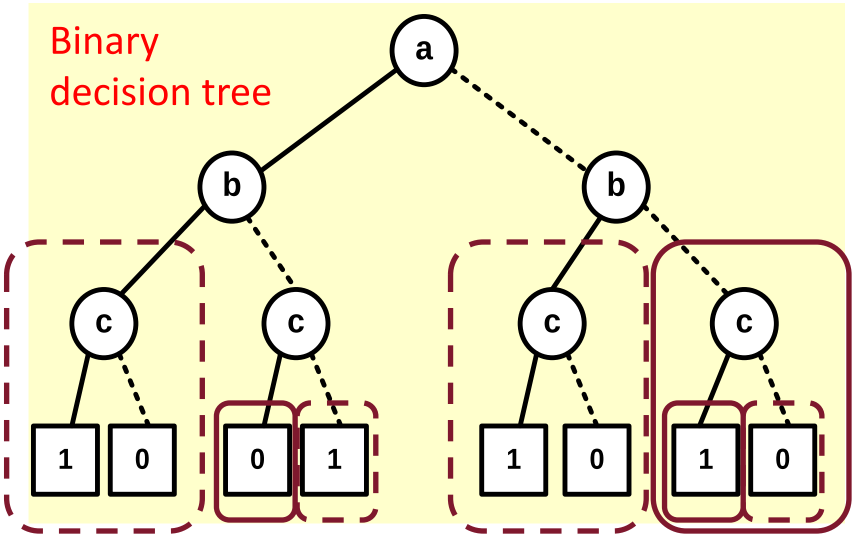
Decision tree for Boolean functions:
Substitution (valuation) of a variable is a decision

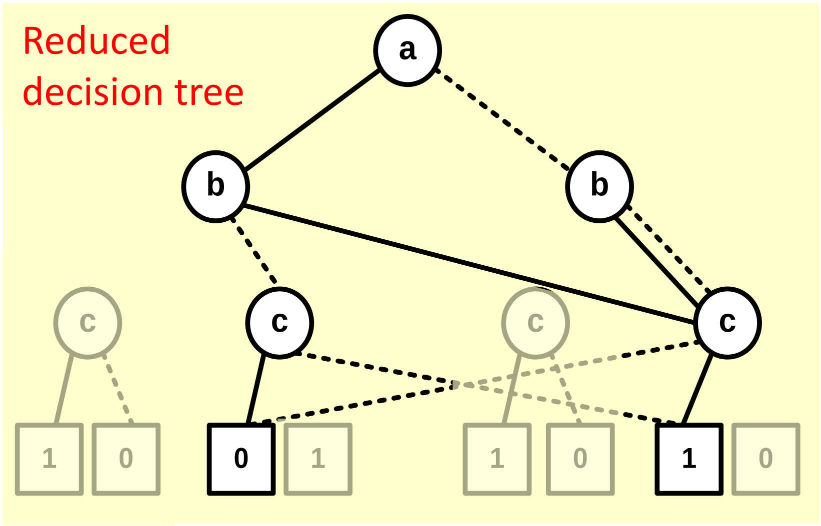
- Example: f(x,y)
- Valuation of all variables results in 1 or 0 in leaf nodes

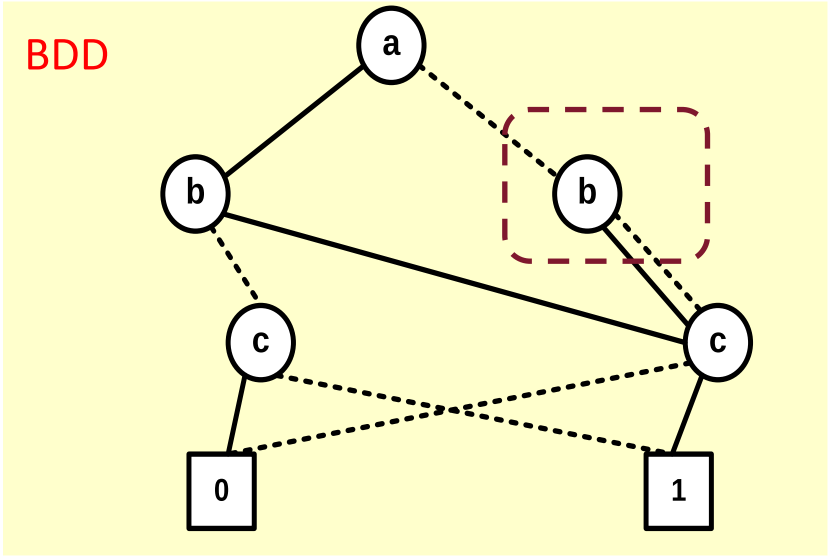


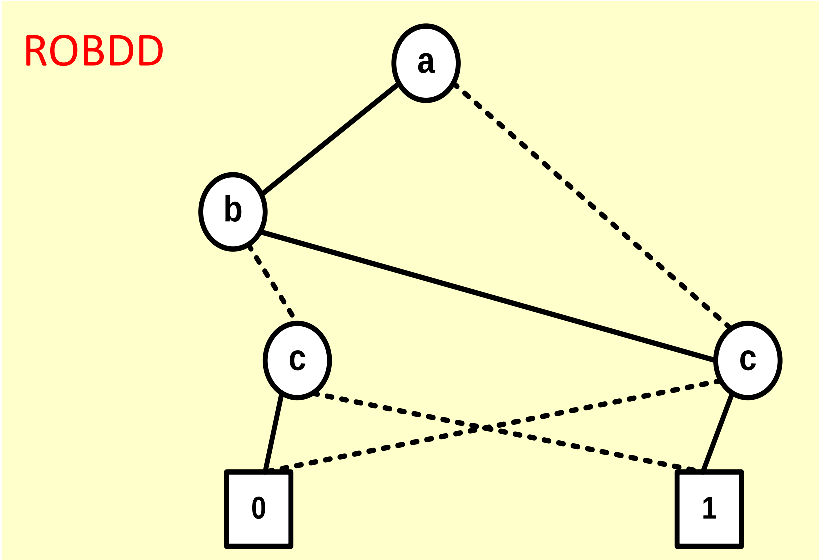
- We get a binary decision diagram (BDD), if we merge all identical subtrees
- We get an ordered binary decision diagram (OBDD), if we substitute the variables in the same order during decomposition
- We get a reduced ordered binary decision diagram (ROBDD), if we remove redundant nodes (where both decisions lead to the same node)

Example: From binary decision tree to ROBDD









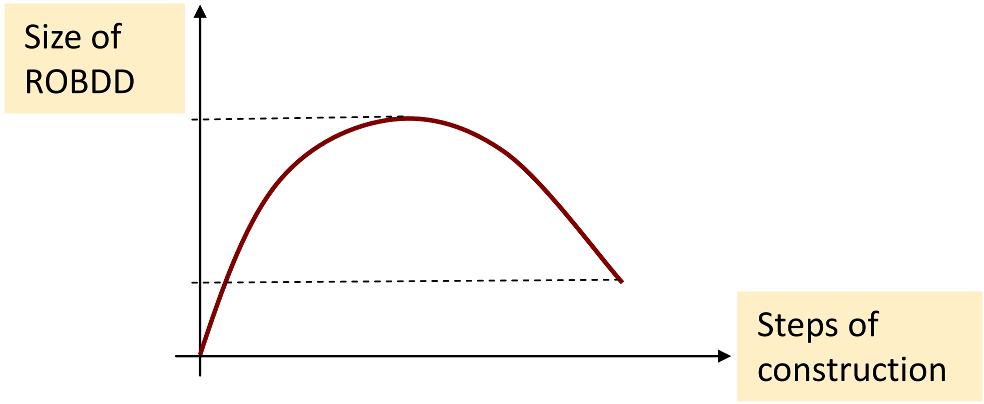
ROBDD properties

- Directed, acyclic graph with one root and two leaves
  - Values of the two leaves are 1 and 0 (true and false)
  - Every node is assigned a test variable
- From every node, two edges leave
  - One for the value 0 (notation: dashed arrow)
  - The other for the value 1 (notation: solid arrow)
- On every path, substituted variables are in the same order
- Isomorphic subgraphs are merged
- Nodes from with both edges would point to the same node are reduced

For a given function, two ROBDDs
with the same variable ordering are isomorphic

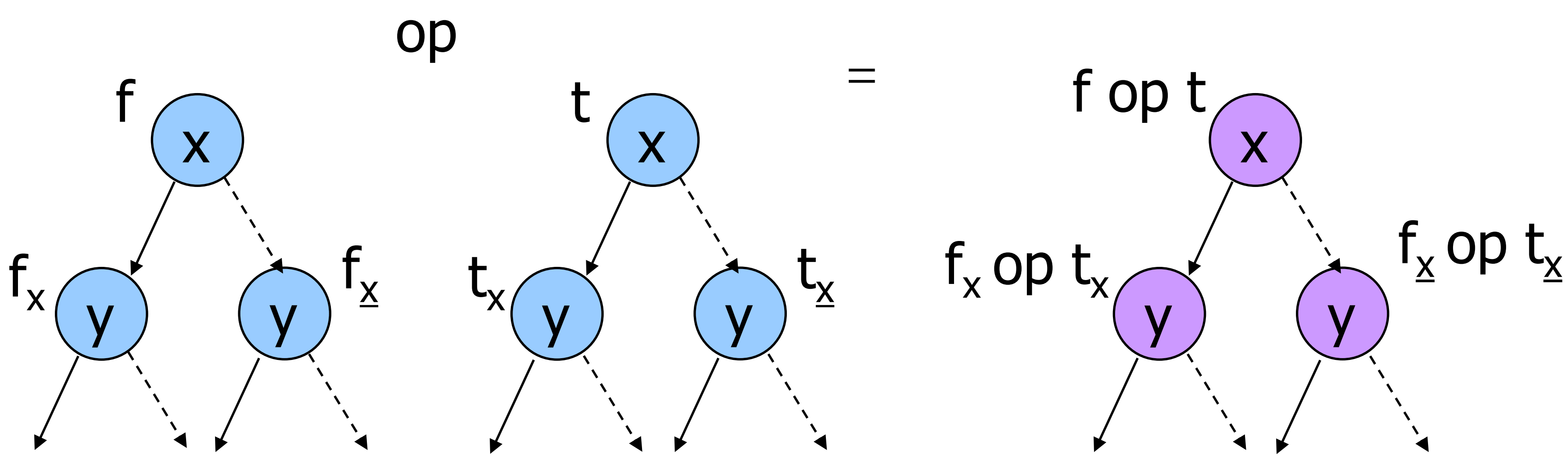
Variable ordering for ROBDDs

- Size of ROBDD
  - For some functions it is very compact
  - For others (such as XOR) it may have an exponential size
- The order of variables has a great impact on the ROBDD size
  - A different order may cause an order of magnitude difference
  - Problem of finding an optimal ordering is NP-complete (→heuristics)
- Memory requirements:
  - If the ROBDD is built by combining functions (e.g., representing product automata), intermediate nodes may appear which can be reduced later



Operations on ROBDDs

- Boolean operators can be evaluated directly on ROBDDs
  - Variables of the functions should be the same in the same order
  - (Recursive) construction of the f op t ROBDD using f and t ROBDDs (here op is a Boolean operator)



Summary: Model checking with ROBDDs

- Implementing model checking:
  - Model checking algorithm: Operations on sets of states (labeling)
  - Symbolic technique: Instead of sets, use Boolean characteristic functions
  - Efficient implementation: Boolean functions handled as ROBDDs
- Benefits
  - ROBDD is a canonical form (equivalence of functions is easy to check)
  - Algorithms can be accelerated (with caching)
  - Reduced storage requirements (depends on variable ordering!)

Dining philosophers:

Number of Philosophers	Size of state space	Number of ROBDD nodes
16	$4,7 \cdot 10^{10}$	747
28	$4,8 \cdot 10^{18}$	1347

Instead of storing 10^{18} states the ROBDD needs ~21kB!

Supplementary material: Construction and operations on ROBDD

Boolean functions as binary decision trees

- Substitution (valuation) of a variable is a decision
- Notation: if-then-else

$$\boxed{x \to f_1, f_0} = (x \wedge f_1) \vee (\neg x \wedge f_0)$$

  - The result is the value of f_1 if x is true (1)
  - The result is the value of f_0 if x is false (0)
  - x is called the test variable, checking its value is a test
- Shannon decomposition of Boolean functions:

$$\left.\begin{array}{r} f = x \to f\,[1/x], f\,[0/x] \\ \text{let } f_x = f\,[1/x]\,;\, f_{\underline{x}} = f\,[0/x] \end{array}\right\} \quad f = x \to f_x\,,\, f_{\underline{x}}$$

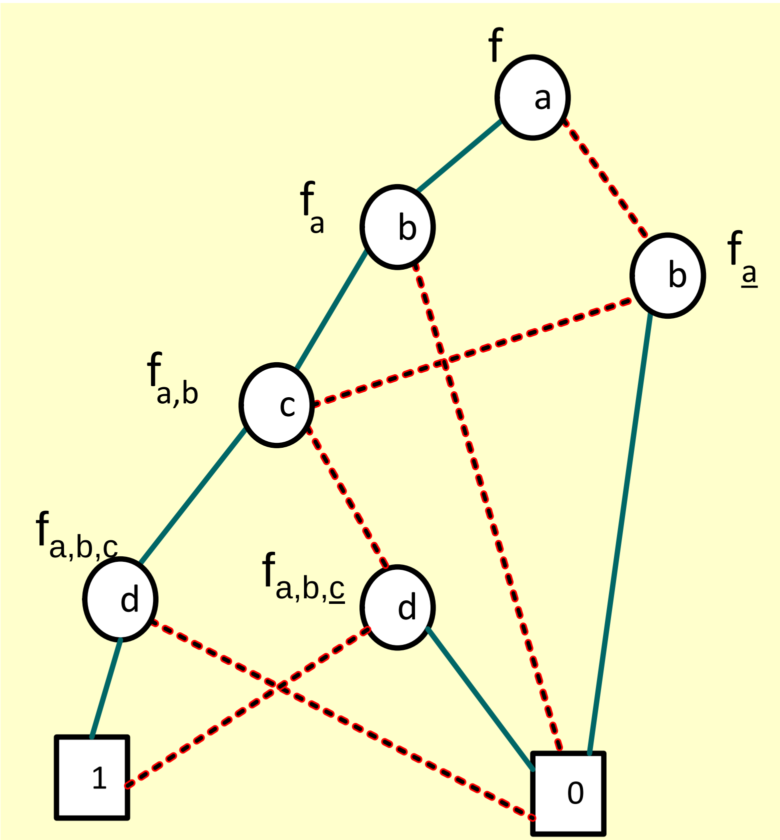
  - The function is decomposed with if-then-else
  - The test variable is substituted, it will not appear in f_x , $f_{\underline{x}}$
  - Repeat until there is a variable left

Example: Manual construction of an ROBDD

Let

$$f = (a \Leftrightarrow b) \wedge (c \Leftrightarrow d)$$

Variable ordering: a, b, c, d



- $f = a \rightarrow f_a,\ f_{\underline{a}}$

  $f_a = (1 \Leftrightarrow b) \wedge (c \Leftrightarrow d)$, $f_{\underline{a}} = (0 \Leftrightarrow b) \wedge (c \Leftrightarrow d)$
- $f_a = b \rightarrow f_{a,b},\ f_{a,\underline{b}}$

  $f_{a,b} = (1 \Leftrightarrow 1) \wedge (c \Leftrightarrow d) = (c \Leftrightarrow d)$

  $f_{a,\underline{b}} = (1 \Leftrightarrow 0) \wedge (c \Leftrightarrow d) = 0$
- $f_{\underline{a}} = b \rightarrow f_{\underline{a},b},\ f_{\underline{a},\underline{b}}$

  $f_{\underline{a},b} = (0 \Leftrightarrow 1) \wedge (c \Leftrightarrow d) = 0$

  $f_{\underline{a},\underline{b}} = (0 \Leftrightarrow 0) \wedge (c \Leftrightarrow d) = (c \Leftrightarrow d)$
- $f_{a,b} = c \rightarrow f_{a,b,c},\ f_{a,b,\underline{c}}$

  $f_{a,b,c} = (1 \Leftrightarrow d)$, $f_{a,b,\underline{c}} = (0 \Leftrightarrow d)$
- $f_{a,b,c} = d \rightarrow f_{a,b,c,d},\ f_{a,b,c,\underline{d}}$

  $f_{a,b,c,d} = (1 \Leftrightarrow 1) = 1$,

  $f_{a,b,c,\underline{d}} = (1 \Leftrightarrow 0) = 0$
- $f_{a,b,\underline{c}} = d \rightarrow f_{a,b,\underline{c},d},\ f_{a,b,\underline{c},\underline{d}}$

  $f_{a,b,\underline{c},d} = (0 \Leftrightarrow 1) = 0$, , $f_{a,b,\underline{c},\underline{d}} = (0 \Leftrightarrow 0) = 1$

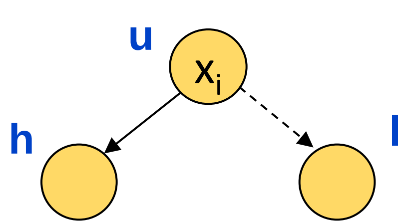
$f_{a,b}$ and $f_{\underline{a},\underline{b}}$ are isomorphic

Storing an ROBDD in memory

- Nodes of the ROBDD are identified by Ids (indices)
- The ROBDD is stored in a table $T: u \rightarrow (i,l,h)$:
  - u: index of node
  - i: index of variable (x_i, $i=1 \ldots n$)
  - l: index of the node reachable through edge corresponding to 0

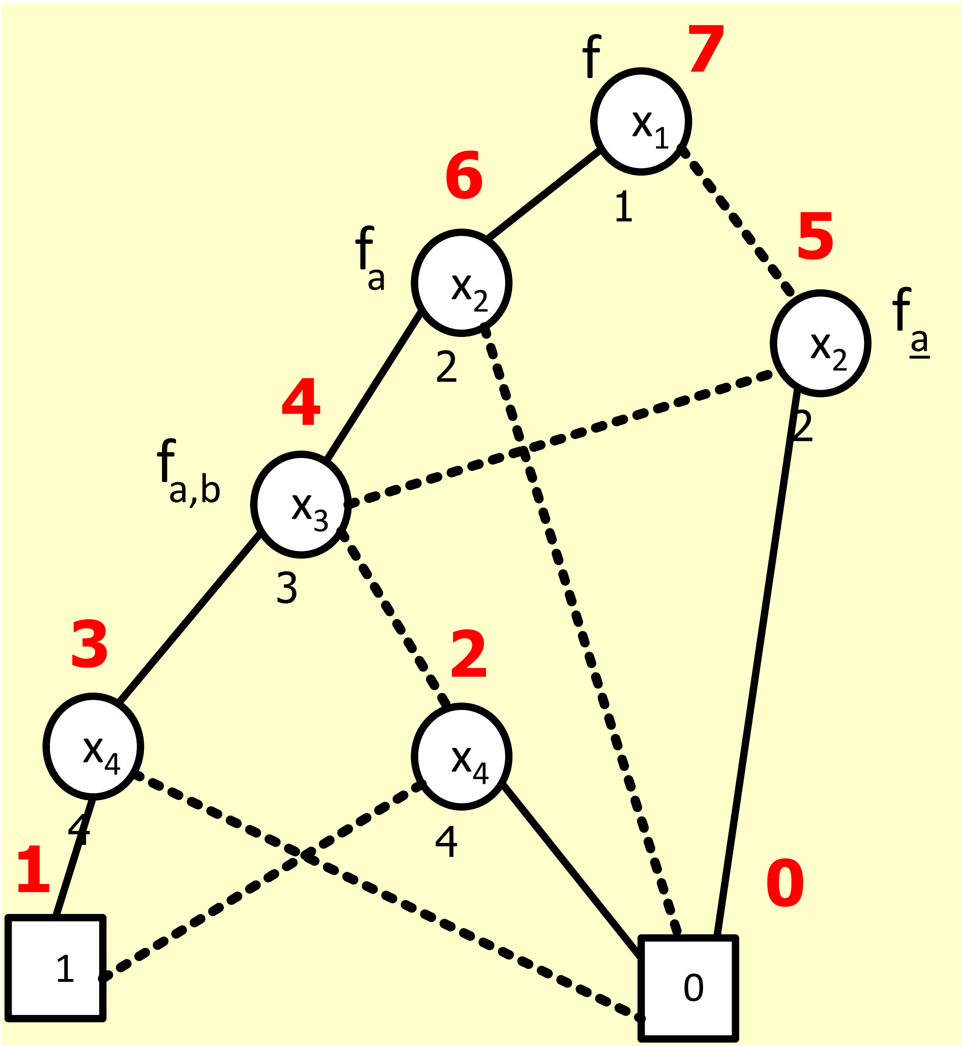
    low
  - h: index of the node reachable through edge corresponding to 1

    high



u	i	l	h
0			
1			
2	4	1	0
3	4	0	1
4	3	2	3
5	2	4	0
6	2	0	4
7	1	5	6

Storing an ROBDD in memory



u	i	l	h
0			
1			
2	4	1	0
3	4	0	1
4	3	2	3
5	2	4	0
6	2	0	4
7	1	5	6

Handling ROBDDs 1.

- Defined operations:
  - init(T)
    - Initializes table T
    - Only the terminal nodes 0 and 1 are in the table
  - add(T,i,l,h):u
    - Creates a new node in T with the provided parameters
    - Returns its index u
  - var(T,u):i
    - Returns from T the index i of the node u
  - low(T,u):l and high(T,u):h
    - Returns the index l (or h) of the node reachable from the node with index u through the edge corresponding to 0 (or 1, respectively)

Handling ROBDDs 2.

- To look up ROBDD nodes, we use another table H: (i,l,h) $\rightarrow$ u
- Operations:
  - init(H)
    - Initializes an empty H
  - member(H,i,l,h):t
    - Checks if the triple (i,l,h) is in H; t is a Boolean value
  - lookup(H,i,l,h):u
    - Looks up the triple (i,l,h) from table H
    - Returns the index u of the matching node
  - insert(H,i,l,h,u)
    - Inserts a new entry into the table

Handling ROBDDs 3.

Creating nodes: Mk(i,l,h)

- Where i is the index of variable, l and h are the branches
- If l=h, i.e. the branches would lead to the same node
  - then we don't need a new node
  - we can return any branches
- If H already contains a triple (i,l,h)
  - then we don't need a new node
  - ⇒ there exists an isomorphic subtree, return that
- If H does not contain such a triple (i,l,h)
  - then we need to create it and return its index

```
Mk(i,l,h){
    if l=h then
        return l;
    else if member(H,i,l,h) then
        return lookup(H,i,l,h);
    else {
        u=add(T,i,l,h);
        insert(H,i,l,h,u);
        return u;
    }
}
```

Handling ROBDDs 4.

Building an ROBDD: Build(f) and Build'(t,i) recursive helper function

```
Build(f) {
   init(T); init(H);
   return Build'(f,1);
}

Build'(t,i){
   if i>n then
         if t==false then return 0 else return 1
   else {v0 = Build'(t[0/x_i],i+1);
         v1 = Build'(t[1/x_i],i+1);
         return Mk(i,v0,v1)}
}
```

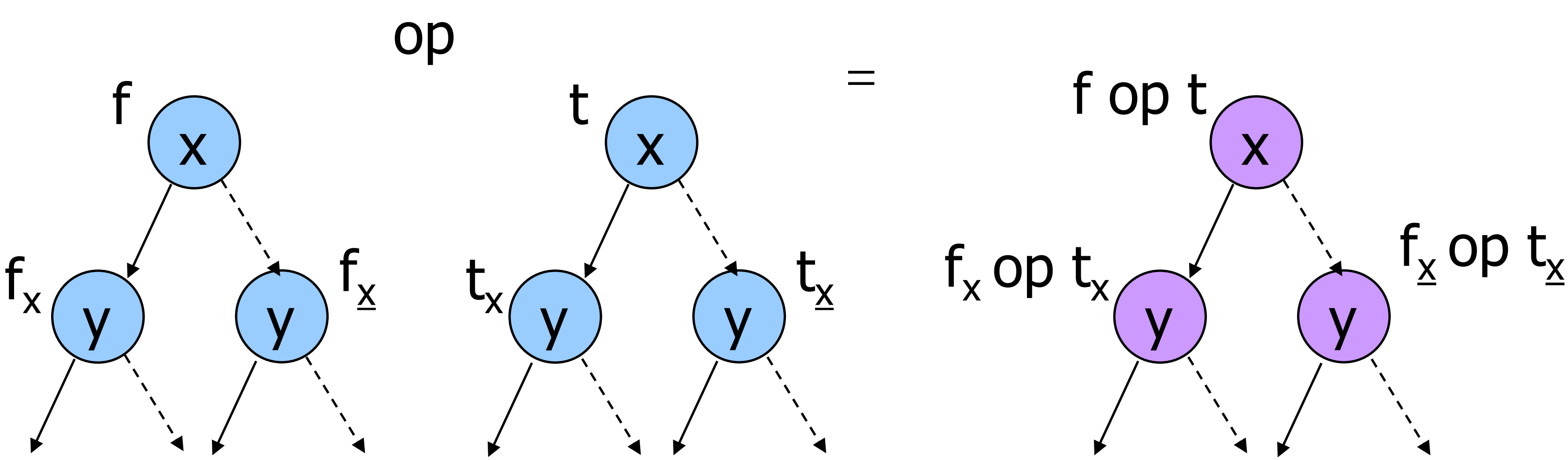
Will traverse variables recursively

Reached a terminal node (every variable substituted)

Recursive building; Mk() will check isomorphic subtrees

- Boolean operators can be evaluated directly on ROBDDs
  - Variables of the functions should be the same and in the same order
- Equivalence for functions f, t (op is a Boolean operator):

1) $f \text{ op } t = (x \rightarrow f_x, f_{\underline{x}}) \text{ op } (x \rightarrow t_x, t_{\underline{x}}) \;=\; x \rightarrow (f_x \text{ op } t_x), (f_{\underline{x}} \text{ op } t_{\underline{x}})$



Operations on ROBDDs (cont'd)

- Boolean operators can be evaluated directly on ROBDDs
  - Variables of the functions should be the same in the same order
- Equivalence for functions f, t (op is a Boolean operator):

1) $f \text{ op } t = (x \rightarrow f_x, f_{\underline{x}}) \text{ op } (x \rightarrow t_x, t_{\underline{x}}) \;=\; x \rightarrow (f_x \text{ op } t_x), (f_{\underline{x}} \text{ op } t_{\underline{x}})$

- Additional rules (in case of missing variables due to reduction):

2) $f \text{ op } t = (x \rightarrow f_x, f_{\underline{x}}) \text{ op } t = x \rightarrow (f_x \text{ op } t), (f_{\underline{x}} \text{ op } t)$

3) $f \text{ op } t = f \text{ op } (x \rightarrow t_x, t_{\underline{x}}) = x \rightarrow (f \text{ op } t_x), (f \text{ op } t_{\underline{x}})$

- Based on these rules App(op,i,j) can be defined recursively
  - where i, j: indices of the root nodes of operands
- Drawback: slow
  - worst-case 2^n exponential

Accelerated operation

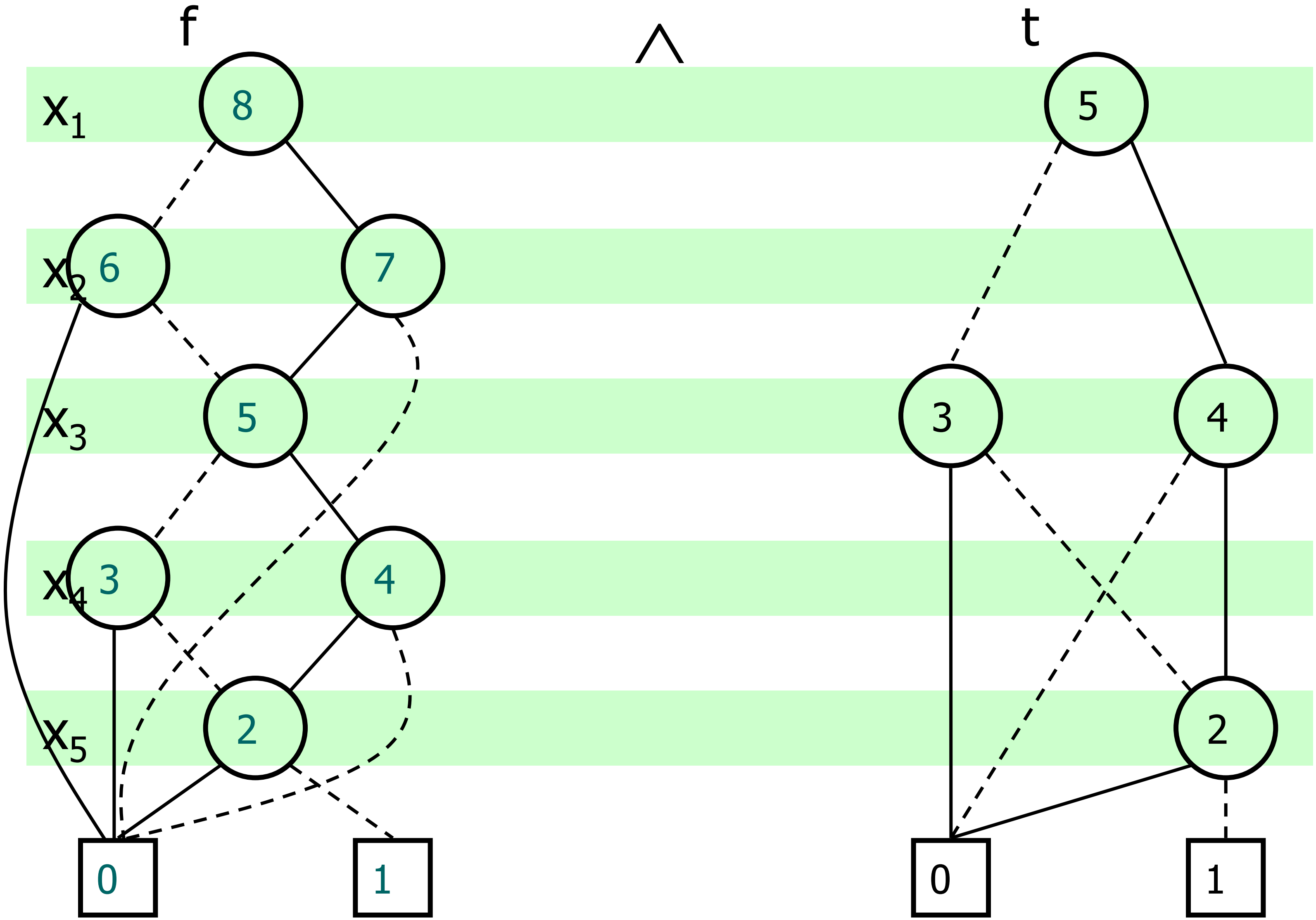
- Let G(op,i,j) be a cache table that contains the results of App(op,i,j) (these are nodes)
- The four cases of the algorithm:
  - Both nodes are terminal: return a terminal based on the Boolean operation (e.g. $0 \wedge 1 = 0$)
  - If the variable indices for both operands are the same, then call App(op,i,j) with the 0 branches and with the 1 branches based on equivalence (1)
  - If one variable index is less, then that node is paired with the 0 and 1 branches of the other node based on rules (2) or (3)

Pseudo-code of the operation

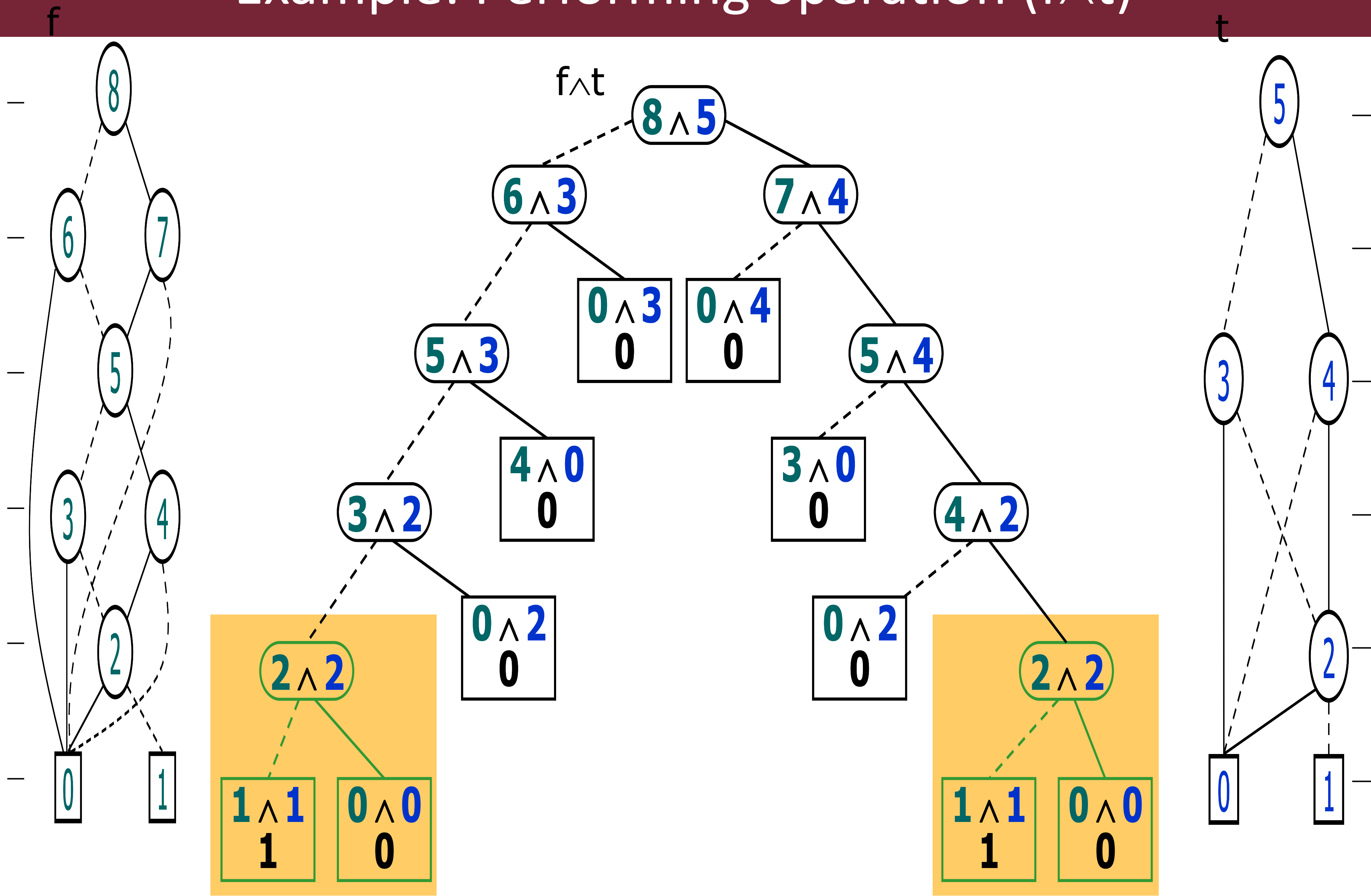
```
Apply(op,f,t){
  init(G);
  return App(op,f,t);
}

App(op,u1,u2) {
   if (G(op,u1,u2) != empty) then return G(op,u1,u2);
   else if (u1 in {0,1} and u2 in {0,1}) then u = op(u1,u2);
   else if (var(u1) = var(u2)) then
        u=Mk(var(u1),App(op,low(u1),low(u2)),
                     App(op,high(u1),high(u2)));
   else if (var(u1) < var(u2)) then
        u=Mk(var(u1), App(op,low(u1),u2),App(op,high(u1),u2));
   else (* if (var(u1) > var(u2)) then *)
        u=Mk(var(u2), App(op,u1,low(u2)),App(op,u1,high(u2)));
  G(op,u1,u2)=u;
  return u;
}
```

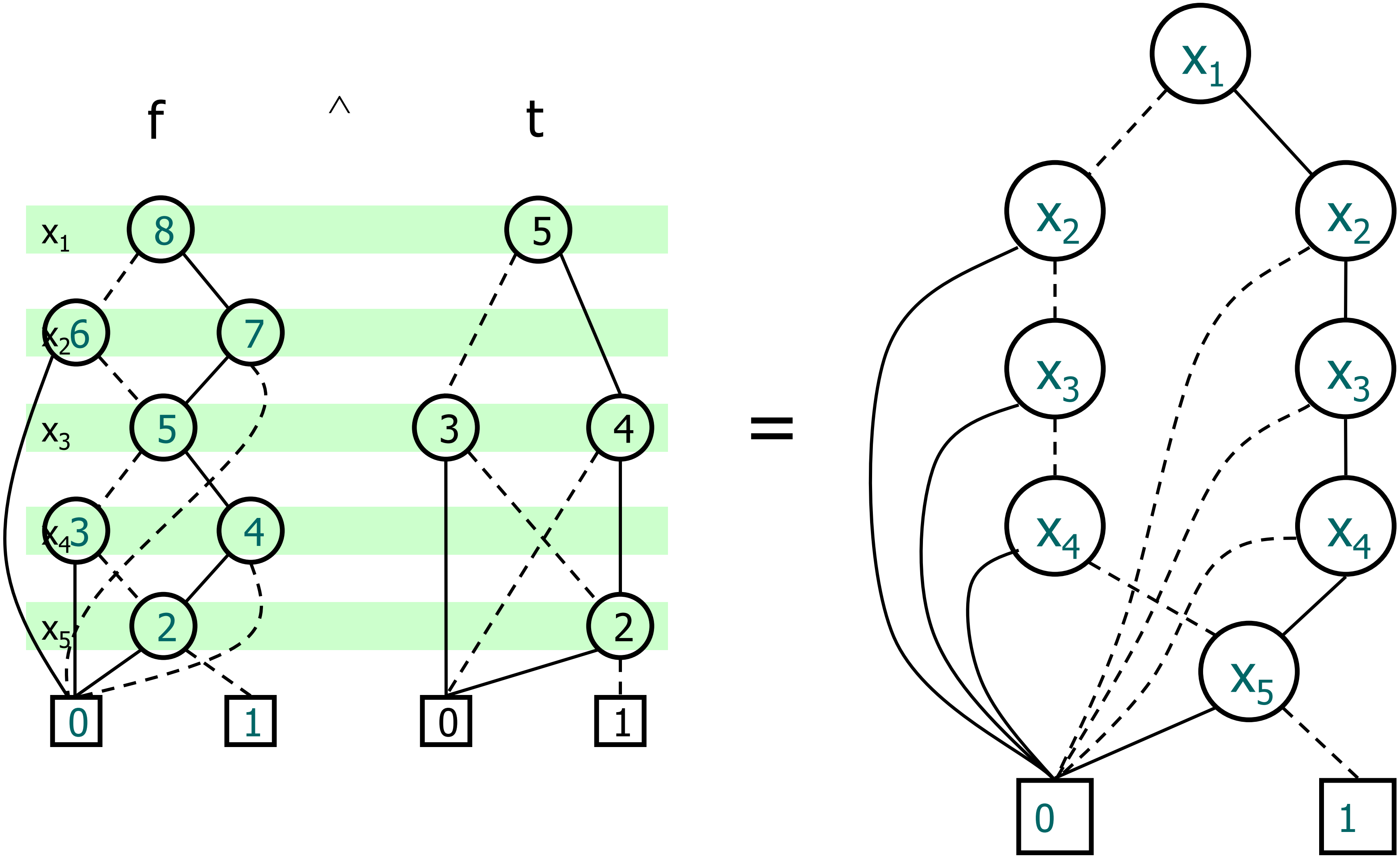
Example: Performing operation (f∧t)



Example: Performing operation (f∧t)



Example: Result of operation (f∧t)



Substiute a variable in an ROBDD

Substitute (bind) variables with constants (e.g. $(\neg x \wedge y)^{[y=1]} = \neg x$):

The value of x_j should be b in the ROBDD rooted in u

```
Restrict(u,j,b) {
  return Res(u,j,b);
}

Res(u,j,b) {
  if var(u) > j then return u;
  else if var(u) < j then
    return Mk(var(u),
              Res(low(u),j,b),
              Res(high(u),j,b));
  else
    if b=0 then
      return Res(low(u),j,b)
    else
      return Res(high(u),j,b);
}
```

If we are lower than the variable to substitute, then the original subtree is returned

If we are higher, then we need recursive building

If we are at the variable to substitute, we process only the branch b